§ 21. The Interaction Lagrangian and the S-Matrix. (Bogoliubov & Shirkov).

21.1 Expansion of the S-Matrix in Powers of the Interaction

Perturbative treatment: We propose an expansion in powers of $g(x)$:

$$S(g) = 1 + \sum_{n\geq 1} \frac{1}{n!} \int S_n(x_1,\dots,x_n)\, g(x_1)\cdots g(x_n)\, dx_1\cdots dx_n .$$

- S_n are "polylocal operators" $A(x_1,\dots,x_n) = \sum K_{\dots\alpha\dots}(x_1,\dots,x_n) : \dots u_\alpha(x_j)\dots :$ ← field operators.

  In the case of fermionic fields, these should appear in pairs.

  $\Rightarrow [A_1(\{x_i\}), A_2(\{y_j\})] = 0$ if $\{x_i\} \sim \{y_j\}$.

- If $g \in \mathcal{S}(\mathbb{R}^4)$, then we have "better chances" of convergence for the individual terms (but the whole series might still be divergent).
- It is clear from its definition that $S_n(x_1,\dots,x_n)$ must be symmetric in its arguments:

  $$S_n(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = S_n(x_1,\dots,x_n) \quad \forall \sigma \in \text{Permutation group of } n \text{ elements.}$$

§ 21.2 Conditions of Covariance, Unitarity and Causality for S_n.

- Covariance: $\quad S(\Lambda\cdot g) = U(\Lambda)\, S(g)\, U(\Lambda)^+$

$$\Rightarrow \int U(\Lambda) S_n(x_1,\dots,x_n) U(\Lambda)^+ g(x_1)\cdots g(x_n)\, dx_1\cdots dx_n = \int S_n(x_1,\dots,x_n)\, g(\Lambda^{-1}x_1)\cdots g(\Lambda^{-1}x_n)\, dx_1\cdots dx_n$$

$$\Rightarrow \quad U(\Lambda) S_n(x_1,\dots,x_n) U(\Lambda)^+ = S_n(\Lambda x_1,\dots,\Lambda x_n).$$

- Unitarity: $\quad S(g)S(g)^+ = \mathbb{1} \quad \Rightarrow \quad$ (taking g to be real)

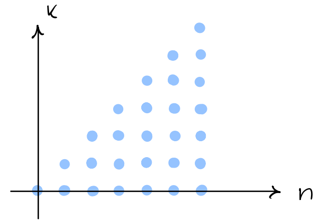
$$\left(\mathbb{1} + \sum_{k\geq 1}\frac{1}{k!}\int S_k(x_1,\dots,x_k)\, g(x_1)\cdots g(x_k)\, dx_1\cdots dx_k\right)\left(\mathbb{1} + \sum_{m\geq 1}\frac{1}{m!}\int S_m^+(x_1,\dots,x_m)\, g(x_1)\cdots g(x_m)\, dx_1\cdots dx_m\right) = \mathbb{1}$$

$\Rightarrow$

$$\sum_{k,m\geq 0} \frac{1}{k!\,m!} \int S_k(x_1,\dots,x_k)\, S_m^+(x_{k+1},\dots,x_{k+m})\, g(x_1)\cdots g(x_k)\cdots g(x_{k+m})\, dx_1\cdots dx_{k+m} = \mathbb{1}$$

Change of variables : $(k,m) \longrightarrow (k,n)$, $n = k+m$.

$(S_0 = \mathbb{1})$



$\Rightarrow$

$$\sum_{0\leq k\leq n} \frac{1}{k!\,(n-k)!} \int S_k(x_1,\dots,x_k)\, S_{n-k}^+(x_{k+1},\dots,x_n)\, g(x_1)\cdots g(x_n)\, dx_1\cdots dx_n = 0 \qquad (21.7)$$

(for $\underline{\underline{n>0}}$)

Since g is arbitrary, we would like to conclude from (21.7) that

$$\sum_{0\leq k\leq n} \frac{1}{k!\,(n-k)!} S_k(x_1,\dots,x_k)\, S_{n-k}^+(x_{k+1},\dots,x_n) \qquad (21.8)$$

is equal to zero. But this is not true, because the product $S_k(\{x\})\, S_{n-k}^+(\{x'\})$ is not symmetric in its argument.

Let discuss this issue in some detail.

If V is a vector space and $\varphi \in V^*$ a dual vector, then it is clear that

$\varphi(v) = 0 \quad \forall\, v \in V$ implies $v = 0$.

For $g \in \mathcal{S}(\mathbb{R})$, we may consider a map of the form

$$\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$$
$$g \longmapsto \langle \ell_F, g\rangle := \int_{\mathbb{R}} F(x)\, g(x)\, dx.$$

If $\langle \ell_F, g\rangle = 0 \quad \forall g = 0$, we would like to conclude that $F = 0$. But here we must be careful, as F could be a locally integrable function with support on a set of Lebesgue measure zero. In that case we will only be able to claim that $[F] = 0$, $[F] \in L^1_{loc}(\mathbb{R})$.

Leaving aside that technical point, let assume that $\int F(x)\, g(x)\, dx = 0$ for "any" function g, where we will also consider $\delta_a(x)$ as one possible choice for g. In that case we have $0 = \int_{\mathbb{R}} F(x)\,\delta_a(x)\,dx = F(a)$. Of course this is just a heuristic argument, but we are fine with it.

Now suppose we are given a function $F(x,y)$ such that

$$\int_{\mathbb{R}^2} F(x,y)\,\varphi(x,y)\,dx\,dy = 0 \qquad \text{"}\forall \varphi\text{"} \qquad (*)$$

The same argument leads to the conclusion that $F=0$.

But what if, instead of $(*)$ above, what we have is

$$\int_{\mathbb{R}^2} F(x,y)\,g(x)\,g(y)\,dx\,dy = 0 \qquad \forall g \quad ?$$

We could think of approximating functions $\varphi(x,y)$ as "sums" of products of the form $g_1(x)\,g_2(x)$, with g_1 and g_2 not necesarilly equal.

Now, let us assume that F is <u>symmetric</u>, $F(x,y) = F(y,x)$. In that case, we can put $g = g_1 + g_2$ and write:

$$0 = \int F(x,y)\,g(x)\,g(y)\,dx\,dy = \int F(x,y)\,\big(g_1(x)+g_2(x)\big)\big(g_1(y)+g_2(y)\big)\,dx\,dy$$

By hypothesis

$$= \cancel{\int F(x,y)\,g_1(x)\,g_1(y)\,dx\,dy}^{\,0} + \int F(x,y)\,\big(g_1(x)\,g_2(y) + g_2(x)\,g_1(y)\big)\,dx\,dy$$

$$+ \cancel{\int F(x,y)\,g_2(x)\,g_2(y)\,dx\,dy}^{\,0}$$

If $F(x,y) = F(y,x)$, we then obtain

$$\int F(x,y)\,g_1(x)\,g_2(y)\,dx\,dy = 0,$$

from which we can now obtain $F=0$.

The idea, then, is to "symmetrize" (21.7). Let us consider a few simple cases:

For $n=2$, (21.7) reads as follows:

$$\sum_{0\le k\le 2}\frac{1}{k!\,(n-k)!}\int S_k(x_1,\dots,x_k)\,\overset{+}{S}_{2-k}(x_{k+1},x_2)\,g(x_1)\,g(x_2)\,dx_1dx_2 =$$

$$= \int\left(\frac{1}{2!}S_0\,\overset{+}{S}_2(x_1,x_2) + S_1(x_1)\,\overset{+}{S}_1(x_2) + S_2(x_1,x_2)\frac{S_0}{2!}\right) g(x_1)\,g(x_2)\,dx_1dx_2.$$

$$= \int\left(\frac{1}{2!}\overset{+}{S}_2(x_1,x_2) + S_1(x_1)\,\overset{+}{S}_1(x_2) + \frac{1}{2!}S_2(x_1,x_2)\right) g(x_1)\,g(x_2)\,dx_1dx_2 \overset{!}{=} 0$$

$\searrow$ Writing

$$\int S_1(x_1)\,\overset{+}{S}_1(x_2)\,g(x_1)\,g(x_2)\,dx_1dx_2 = \frac{1}{2}\int S_1(x_1)\,\overset{+}{S}_1(x_2)\big(g(x_1)\,g(x_2) + g(x_2)\,g(x_1)\big)\,dx_1dx_2$$

$$= \frac{1}{2!}\int\big(S_1(x_1)\,\overset{+}{S}_1(x_2) + S_1(x_2)\,\overset{+}{S}_1(x_1)\big)\,g(x_1)\,g(x_2)\,dx_1dx_2$$

Defining a "symmetrization" operator $P\left(\frac{x_1}{x_2}\right)$ as follows,

$$P\left(\frac{x_1}{x_2}\right) S_1(x_1)\,\overset{+}{S}_1(x_2) = S_1(x_1)\,\overset{+}{S}_1(x_2) + S_1(x_2)\,\overset{+}{S}_1(x_1),$$

or more generally $P\left(\frac{x}{y}\right) A(x)\,B(y) = A(x)\,B(y) + B(y)\,A(x)$,

we obtain

$$2!\int\left(\frac{1}{2!}\overset{+}{S}_2(x_1,x_2) + S_1(x_1)\,\overset{+}{S}_1(x_2) + \frac{1}{2!}S_2(x_1,x_2)\right) g(x_1)\,g(x_2)\,dx_1dx_2 =$$

$$= \int\left(\frac{1}{\cancel{2!}}\cancel{2!}\,\overset{+}{S}_2(x_1,x_2) + P\left(\frac{x_1}{x_2}\right) S_1(x_1)\,\overset{+}{S}_1(x_2) + \frac{1}{\cancel{2!}}\cancel{2!}\,S_2(x_1,x_2)\right) g(x_1)\,g(x_2)\,dx_1dx_2$$

$\longrightarrow$ The expression in parentheses is now totally symmetric, and therefore must vanish

$$\Rightarrow \quad \boxed{S_2(x,y) + \overset{+}{S}_2(x,y) + S_1(x)\,\overset{+}{S}_1(y) + S_1(y)\,\overset{+}{S}_1(x) = 0.}$$

Let us now consider the next case, $n=3$. Eq. (21.7) now reads:

$$\int \left(\frac{1}{3!} S_3^+(x_1,x_2,x_3) + \frac{1}{2!} S_1(x_1) S_2^+(x_2,x_3) + \frac{1}{2!} S_2(x_1,x_2) S_1^+(x_3) + \frac{1}{3!} S_3(x_1,x_2,x_3) \right) g(x_1) g(x_2) g(x_3) dx_1 dx_2 dx_3$$

Now, for an expression of the form $A(x) B(y,z)$, we have:

$$3 \int A(x) B(y,z) g(x) g(y) g(z) dx dy dz =$$

$$= \int A(x) B(y,z) \left(g(x) g(y) g(z) + g(y) g(x) g(z) + g(z) g(x) g(y) \right) dx dy dz$$

$$= \int \left(A(x) B(y,z) + A(y) B(x,z) + A(z) B(x,y) \right) dx dy dz$$

Defining a new "symmetrizer" (assuming $B(a,b) = B(b,a)$),

$$P\left(\frac{x_1}{x_2,x_3}\right) A(x_1) B(x_2,x_3) = A(x_1) B(x_2,x_3) + A(x_2) B(x_1,x_3) + A(x_3) B(x_1,x_2),$$

$$\Rightarrow \binom{3}{2} \int A(x) B(x,y) g(x) g(y) g(z) dx dy dz = \int P\left(\frac{x}{y,z}\right) A(x) B(y,z) g(x) g(y) g(z) dx dy dz$$

as well as

$$P\left(\frac{x_1,x_2}{x_3}\right) A(x_1,x_2) B(x_3) = A(x_1,x_2) B(x_3) + A(x_1,x_3) B(x_2) + A(x_2,x_3) B(x_1),$$

we obtain

$$0 = 3! \int \left(\frac{1}{3!} S_3^+(x_1,x_2,x_3) + \frac{1}{2!} S_1(x_1) S_2^+(x_2,x_3) + \frac{1}{2!} S_2(x_1,x_2) S_1^+(x_3) + \frac{1}{3!} S_3(x_1,x_2,x_3) \right) g(x_1) g(x_2) g(x_3) dx_1 dx_2 dx_3 =$$

$$= \int \left(\frac{\cancel{3!}}{\cancel{3!}} S_3^+(x_1,x_2,x_3) + \frac{3!}{2!} S_1(x_1) S_2^+(x_2,x_3) + \frac{3!}{2!} S_2(x_1,x_2) S_1^+(x_3) + \frac{\cancel{3!}}{\cancel{3!}} S_3(x_1,x_2,x_3) \right) g(x_1) g(x_2) g(x_3) dx_1 dx_2 dx_3$$

$$= \int \left(S_3^+(x_1,x_2,x_3) + \binom{3}{2} S_1(x_1) S_2^+(x_2,x_3) + \binom{3}{2} S_2(x_1,x_2) S_1^+(x_3) + S_3(x_1,x_2,x_3) \right) g(x_1) g(x_2) g(x_3) dx_1 dx_2 dx_3$$

$$= \int \left(S_3^+(x_1,x_2,x_3) + P\left(\frac{x_1}{x_2,x_3}\right) S_1(x_1) S_2^+(x_2,x_3) + P\left(\frac{x_1,x_2}{x_3}\right) S_2(x_1,x_2) S_1^+(x_3) + S_3(x_1,x_2,x_3) \right) g(x_1) g(x_2) g(x_3) dx_1 dx_2 dx_3$$

$$\Rightarrow \quad S_3(x_1,x_2,x_3) + S_3^+(x_1,x_2,x_3) + P\left(\frac{x_1}{x_2,x_3}\right) S_1(x_1)\, S_2^+(x_2,x_3) + P\left(\frac{x_1,x_2}{x_3}\right) S_2(x_1,x_2)\, S_1^+(x_3) = 0.$$

The definition of the "symmetrizer" symbol P is now clear:

$P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right)$ denotes the sum over all the $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ ways of dividing the set of points $x_1,\dots,x_n$ into two sets of k and $(n-k)$ points. Permutations within each of these two sets are not taken into account, since the functions S_k are symmetric in their arguments.

We therefore conclude that the following identity must hold:

$$\boxed{S_n(x_1,\dots,x_n) + S_n^+(x_1,\dots,x_n) + \sum_{1\leq k\leq n-1} P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_k(x_1,\dots,x_k)\, S_{n-k}^+(x_{k+1},\dots,x_n) = 0} \qquad (21.9)$$

Now, let us return to the (differential form of the) causality condition.

Unitarity implies

$$0 = \frac{\delta}{\delta g(y)}\left(SS^+\right) = \frac{\delta S}{\delta g(y)} S^+ + S\frac{\delta S^+}{\delta g(y)}$$

$$\Rightarrow \quad \left(\frac{\delta S}{\delta g(y)} S^+\right)^+ = S\,\frac{\delta S^+}{\delta g(y)} = -\frac{\delta S}{\delta g(y)} S^+ \qquad \text{(anti-hermitean)}$$

It is better to consider the following hermitean operator:

$$H(y;g) = i\,\frac{\delta S(g)}{\delta g(y)}\, S^+(g) \qquad (21.10)$$

Let us compute the variaton of $S(g)$ in the perturbative expansion

$n=1$: $\quad \frac{\delta}{\delta g(y)} \int S_n(x)\, g(x)\, dx = S_n(y)$

$$\delta I[g] = I[g+\delta g] - I[g]$$

$n=2$: $$\frac{\delta}{\delta g(y)} \frac{1}{2!} \int S_2(x_1,x_2)\, g(x_1) g(x_2)\, dx_1 dx_2$$

$$\delta \int S_2(x_1,x_2)\, g(x_1) g(x_2)\, dx_1 dx_2 =$$

$$= \int S_2(x_1,x_2)\big(g(x_1)+\delta g(x_1)\big)\big(g(x_2)+\delta g(x_2)\big)\, dx_1 dx_2 - \int S_2(x_1,x_2)\, g(x_1) g(x_2)\, dx_1 dx_2$$

$$\simeq \int S_2(x_1,x_2)\big(g(x_1)\,\delta g(x_2) + g(x_2)\,\delta g(x_1)\big)\, dx_1 dx_2$$

2 order terms are dropped

$$= \int\int \big(S_2(x_1,x_2) + S_2(x_2,x_1)\big)\, g(x_1)\, \delta g(x_2)\, dx_1 dx_2$$

$$= 2 \int \left[\int S_2(x,y)\, g(x)\, dx \right] \delta g(y)\, dy$$

$$\Rightarrow \frac{1}{2!} \frac{\delta}{\delta g(y)} \left[\int S_2(x_1,x_2)\, g(x_1) g(x_2)\, dx_1 dx_2 \right] = \int S_2(x,y)\, g(x)\, dx$$

$n=3$: $$\frac{\delta}{\delta g(y)} \left[\frac{1}{3!} \int S_3(x_1,x_2,x_3)\, g(x_1) g(x_2) g(x_3)\, dx_1 dx_2 dx_3 \right] = ?$$

With $I[g] =$ " $\frac{1}{3!} \int S_3\, g g g$ " we have

$$I[g+\delta g] - I[g] = \frac{1}{3!} \int S_3\, (g+\delta g)(g+\delta g)(g+\delta g) - \frac{1}{3!} \int S_3\, g g g$$

symmetric in (x_1,x_2,x_3)

$$= \frac{1}{3!} \cdot 3 \cdot \int S_3\, g\, g\, \delta g \equiv \frac{1}{2!} \int S_3(x_1,x_2,x_3)\, g(x_1)\, g(x_2)\, \delta g(x_3)\, dx_1 dx_2 dx_3$$

$$\Rightarrow \frac{\delta}{\delta g(y)} \left[\frac{1}{3!} \int S_3(x_1,x_2,x_3)\, g(x_1) g(x_2) g(x_3)\, dx_1 dx_2 dx_3 \right] = \frac{1}{2!} \int S_3(x_1,x_2,y)\, g(x_1) g(x_2)\, dx_1 dx_2$$

It is clear that the formula for the general case is the following one:

$$\frac{\delta}{\delta g(y)} \left[\frac{1}{n!} \int S_n(x_1,x_2,\dots,x_n)\, g(x_1) g(x_2) \cdots g(x_n)\, dx_1 \cdots dx_n \right] = \frac{1}{(n-1)!} \int S_n(x_1,x_2,\dots,x_{n-1},y)\, g(x_1) \cdots g(x_{n-1})\, dx_1 \cdots dx_{n-1}$$

$$\equiv \frac{1}{(n-1)!} \int S_n(y,x_1,x_2,\dots,x_{n-1})\, g(x_1) \cdots g(x_{n-1})\, dx_1 \cdots dx_{n-1}$$

We can now obtain a perturbative expression for $H(y;g) \longrightarrow$

$$H(y;g) = i\,\frac{\delta S(g)}{\delta g(y)}\,S^{+}(g) = i\,\frac{\delta}{\delta g(y)}\left(\sum_{n\geq 0}\frac{1}{n!}\int S_n(x_1,\dots,x_n)\,g(x_1)\cdots g(x_n)\,dx_1\cdots dx_n\right)\times$$

$$\times\left(\sum_{m\geq 0}\frac{1}{m!}\int S^{+}_m(z_1,z_2,\dots,z_m)\,g(z_1)\cdots g(z_m)\,dz_1\cdots dz_m\right) =$$

$$= \left(\sum_{n\geq 1}\frac{i}{(n-1)!}\int S_n(y,x_1,\dots,x_{n-1})\,g(x_1)\cdots g(x_{n-1})\,dx_1\cdots dx_{n-1}\right)\left(\sum_{m\geq 0}\frac{1}{m!}\int S^{+}_m(z_1,z_2,\dots,z_m)\,g(z_1)\cdots g(z_m)\,dz_1\cdots dz_m\right)$$

because $S_0 = \mathbb{1}$ $\qquad n' = n-1$

$$= \left(\sum_{n'\geq 0}\frac{i}{n'!}\int S_{n'+1}(y,x_1,\dots,x_{n'})\,g(x_1)\cdots g(x_{n'})\,dx_1\cdots dx_{n'}\right)\left(\sum_{m\geq 0}\frac{1}{m!}\int S^{+}_m(z_1,z_2,\dots,z_m)\,g(z_1)\cdots g(z_m)\,dz_1\cdots dz_m\right)$$

$$= \sum_{n\geq 0}{}^{'}\sum_{m\geq 0}\frac{i}{n!\,m!}\int S_{n+1}(y,x_1,\dots,x_n)\,g(x_1,\dots,x_n)\,dx_1\cdots dx_n\int S^{+}_m(x_{n+1},\dots,x_{n+m})\,g(x_{n+1})\cdots g(x_{n+m})\,dx_{n+1}\cdots dx_{n+m}$$

this relabelling does no harm, as everytime the sum over m is performed at fixed n.

$$= \sum_{n=0}^{\infty}\left(\underbrace{\sum_{k=0}^{n}\frac{i}{k!\,(n-k)!}\int S_{k+1}(y,x_1,\dots,x_k)\,S^{+}_{n-k}(x_{k+1},\dots,x_n)\,g(x_1)\cdots g(x_n)\,dx_1\cdots dx_n}\right)$$

$$= \frac{i}{n!}\sum_{0\leq k\leq n}\int P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_{k+1}(y,x_1,\dots,x_k)\,S^{+}_{n-k}(x_{k+1},\dots,x_n)\,g(x_1)\cdots g(x_n)\,dx_1\cdots dx_n$$

(exactly the same computation as before)

$$= i\sum_{n\geq 0}\frac{1}{n!}\int\Big[\underbrace{i\sum_{0\leq k\leq n}P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_{k+1}(y,x_1,\dots,x_k)\,S^{+}_{n-k}(x_{k+1},\dots,x_n)}_{=:\,H_n(y;x_1,\dots,x_n)}\Big]\,g(x_1)\cdots g(x_n)\,dx_1\cdots dx_n$$

Summarizing, if we define

$$H_n(y;x_1,\dots,x_n) := i\sum_{0\leq k\leq n}P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_{k+1}(y,x_1,\dots,x_k)\,S^{+}_{n-k}(x_{k+1},\dots,x_n)\,, \qquad (21.12)$$

then

$$\boxed{H(y;g) = i\,\frac{\delta S(g)}{\delta g(y)}\,S^{+}(g) = \sum_{n\geq 0}\frac{1}{n!}\int H_n(y,x_1,\dots,x_n)\,g(x_1)\cdots g(x_n)\,dx_1\cdots dx_n} \qquad (21.11)$$

The next step consists in evaluating the variational derivative of $H(y;g)$ with respect to "$g(x)$" $\rightarrow$

$$i \frac{\delta}{\delta g(x)} \left(\frac{\delta S(g)}{\delta g(y)} S^{+}(g) \right) = \frac{\delta}{\delta g(x)} H(y;g)$$

$$= \sum_{n \geq 0} \frac{1}{n!} \frac{\delta}{\delta g(x)} \int H_n(y, x_1, \dots, x_n) \, g(x_1) \cdots g(x_n) \, dx_1 \cdots dx_n$$

$$= \sum_{n \geq 1} \frac{1}{(n-1)!} \int H_n(y, x, x_1, \dots, x_{n-1}) \, g(x_1) \cdots g(x_{n-1}) \, dx_1 \cdots dx_{n-1}$$

Since we have arranged all terms in such a way that $H_n(y, x_1, x_2, \dots, x_n)$ is symmetric w.r.t. $x_1, \dots, x_n$, we obtain, from the differential condition of causality,

$$\boxed{H_n(y, x_1, x_2, \dots, x_n) = 0 \quad \text{if} \quad y \gtrsim x_i \text{ for at least one } x_j \; (j = 1, \dots, n)} \qquad (21.13)$$

21.3 Explicit form of $S_1(x)$ and $S_2(x,y)$

So far we have obtained the following identities, expressing at a "perturbative" level the conditions of unitarity, covariance and causality:

- Covariance: $$S_n(x_1, \dots, x_n) = U(\Lambda) \, S_n(\Lambda^{-1}x_1, \dots, \Lambda^{-1}x_n) \, U(\Lambda)^{+} \qquad (a)$$

- Unitarity: $$S_n(x_1, \dots, x_n) + S_n^{+}(x_1, \dots, x_n) + \sum_{1 \leq k \leq n-1} P\begin{pmatrix} x_1, \dots, x_k \\ x_{k+1}, \dots, x_n \end{pmatrix} S_k(x_1, \dots, x_k) \, S_{n-k}^{+}(x_{k+1}, \dots, x_n) = 0 \qquad (b)$$

- Causality: $$H_n(y, x_1, x_2, \dots, x_n) \equiv i \sum_{0 \leq k \leq n} P\begin{pmatrix} x_1, \dots, x_k \\ x_{k+1}, \dots, x_n \end{pmatrix} S_{k+1}(y, x_1, \dots, x_k) \, S_{n-k}^{+}(x_{k+1}, \dots, x_n) = 0$$
  if $y \gtrsim x_i$ for at least one $x_j \; (j = 1, \dots, n)$ $\qquad (c)$

Adding additional physical requirements related to the "correspondence principle", we will proceed to determine the explicit form of S_n. We will start with S_1 and S_2.

Notice that (b) and (c) above can be regarded as recurrence relations, and as such they can in principle be used to determine S_n in terms of S_k $(k \leq n-1)$. (b) will determine the hermitean part of S_n, whereas (c) will determine its anti-hermitean part.

In order to start the induction algorithm, we need to fix $S_1(x)$. This will be done by appealing to the correspondence principle.

Let us now consider the causal factorization property

$$\mathrm{Supp}\, g_2 > \mathrm{Supp}\, g_1 \quad \Rightarrow \quad S(g_1+g_2) = S(g_2)\, S(g_1)$$

If the supports of g_1 and g_2 are space-like, $g_1 \sim g_2$, then we must have $[S(g_1), S(g_2)] = 0$.

Using the series expansion at order 1, this gives

$$\int [S_1(x), S_1(y)]\, g_1(x)\, g_2(y)\, dx\, dy \overset{\text{to this order}}{=} 0$$

It seems, therefore, that assuming (micro-) locality for $S_1(x)$ is a reasonable assumption (more on this condition later) $\longrightarrow$ So we assume the following locality property:

$$x \sim y \quad \Rightarrow \quad [S_1(x), S_1(y)] = 0. \qquad (21.14)$$

From (b), i.e., unitarity, we obtain, for $n=1$, $S_1(x) + S_1^+(x) = 0$. This means that S_1 is anti-hermitean. In other words, unitarity implies that S_1 is of the form

$$S_1(x) = i\, \Lambda_1(x), \quad \text{with } \Lambda_1(x) \text{ hermitean} \qquad (21.16)$$

Finally, covariance implies $S_1(x) = U(\Lambda)\, S_1(\Lambda^{-1}x)\, U(\Lambda)^{\dagger}$

$\Rightarrow$ $\Lambda_1(x)$ must be an Hermitean relativistically covariant operator satisfying the condition of locality.

It turns that, from a physical point of view, $\Lambda_1(x)$ can be regarded as the interaction Lagrangian.

Consider the classical action I for the case where the interaction is "switched on/off" by means of a test function g:

$$I = \int \mathcal{L}_0(x)\,dx + \int \mathcal{L}(x)\,g(x)\,dx$$

$\mathcal{L}_0$: free Lagrangian

$\mathcal{L}$: interaction Lagrangian.

→ Regard $g(x)$ as an "infinitesimal of 1st order"

↳ The action I will be altered by $\int \mathcal{L} g$, where $\mathcal{L}$ depends on the <u>free</u> field functions.

Now, Schrödinger dynamics → $i\partial_t \Psi = H\Psi$

Wave function in terms of action → $\Psi = e^{iI}$

Free case: $\Psi_0 = e^{iI_0}$ → Interacting case: $\Psi = e^{iI} = e^{i\int \mathcal{L}(x) g(x) dx}\, \Psi_0$

For an infinitesimal change, $\Psi \to \Psi' = \Psi + \delta\Psi$, $\delta\Psi = \left(i \int \mathcal{L}(x) g(x)\, dx\right)\Psi$

Now "invoke" a "correspondence principle" and demand

$$|\Phi\rangle \to |\Phi'\rangle = |\Phi\rangle + |\delta\Phi\rangle, \quad |\delta\Phi\rangle = \left(i\int \mathcal{L}(x) g(x)\, dx\right)|\Phi\rangle,$$

i.e., for infinitesimal g, $S(g)$ should be of the form

$$S(g) \approx 1 + i\int \mathcal{L}(x)\, g(x)\, dx,$$

i.e., $\mathcal{L}_{int} \equiv \Lambda_1$

↳

$$S_1(x) = i\,\mathcal{L}_{int}(x) \qquad (21.20)$$

↳ And so, the interaction Lagrangian must be a local, Hermitian, and relativistically covariant combination of operator field functions.

→ <u>Determination of $S_2(x,y)$</u>

The causality condition, (21.13), states that

$$H_n(y, x_1, x_2, .., x_n) = 0 \quad \text{if} \quad y \gtrsim x_i \text{ for at least one } x_j \ (j=1,\dots,n),$$

where H_n has been defined in (21.11) as

$$H_n(y; x_1,\dots,x_n) := i \sum_{0\le k\le n} P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_{k+1}(y, x_1,\dots,x_k)\, S^+_{n-k}(x_{k+1},\dots,x_n),$$

$$S_{0+1} S^+_{1-0} + S_2 S^+_0$$

In the particular case $n=1$, we have:

$$H_1(y; x_1) = i \underbrace{P\left(\frac{"\phi"}{x_1}\right)}_{=id} S_1(y) S^+_1(x_1) + i \underbrace{P\left(\frac{x_1}{"\phi"}\right)}_{=id} S_2(y, x_1) \underbrace{S^+_0}_{=1}$$

↘

$$H_1(x,y) = i S_2(x,y) + i S_1(x) S^+_1(y) \qquad S_1 = i\mathcal{L} \to S^+_1 = -i\mathcal{L}^+$$

$$= i S_2(x,y) + i \mathcal{L}(x)\mathcal{L}(y)$$

Now, the (differential) condition of causality implies that $H_2(x,y) = 0$ whenever $x \gtrsim y$:

$$" x \gtrsim y \Rightarrow S_2(x,y) = -\mathcal{L}(x)\mathcal{L}(y)" \qquad (*)$$

If $x \lesssim y$, then we have:

$$S_2(x,y) \underset{\substack{\uparrow \\ S_2 \text{ is symmetric.}}}{=} S_2(y,x) \overset{(*)}{=} -\mathcal{L}(y)\mathcal{L}(x)$$

Thus, we conclude that

$$S_2(x,y) = \begin{cases} -\mathcal{L}(x)\mathcal{L}(y), & x \gtrsim y \\ -\mathcal{L}(y)\mathcal{L}(x), & x \lesssim y \end{cases} \qquad (21.25)$$

Notice, in particular, that if $x \sim y$, then we have two equivalent expressions for S_2 → $x \sim y \Rightarrow S_2(x,y) = -\mathcal{L}(x)\mathcal{L}(y) \overset{!}{=} -\mathcal{L}(y)\mathcal{L}(x)$, i.e.,

$$x \sim y \Rightarrow [\mathcal{L}(x), \mathcal{L}(y)] = 0.$$

→ We see that the locality condition for the interaction Lagrangian follows from the causality condition on $S(g)$.

As a consistency check, let us verify that the unitarity condition is satisfied:
↳ this must be done, as causality and unitarity are independent conditions!

For $x \gtrsim y$ we have

$$S_2(x,y) + S_2^+(x,y) + S_1(x)\,S_1^+(y) + S_1(y)\,S_1^+(x) =$$

$$= -\mathcal{L}(x)\mathcal{L}(y) - \left(\mathcal{L}(x)\mathcal{L}(y)\right)^+ + \mathcal{L}(x)\mathcal{L}(y) + \mathcal{L}(y)\mathcal{L}(x)$$

$$= -\mathcal{L}(x)\mathcal{L}(y) - \mathcal{L}(y)\mathcal{L}(x) + \mathcal{L}(x)\mathcal{L}(y) + \mathcal{L}(y)\mathcal{L}(x)$$

$$= 0.$$

The same identity is obtained if we assume $x \lesssim y$.

The above result can be expressed in terms of a time-ordered product:

$$S_2(x,y) = -T\left(\mathcal{L}(x)\mathcal{L}(y)\right). \tag{21.29}$$

Notice, however, that there is an ambiguity when $x^0 = y^0$.

§ 21.5 Determination of the functions S_n for arbitrary n.

Going back, once again, to equs. (21.11) and (21.13) we have, for $n=2$,

$$H_2(y;x_1,x_2) := i \sum_{0 \le k \le 2} P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_2}\right) S_{k+1}(y,x_1,\dots,x_k)\, S_{2-k}^+(x_{k+1},\dots,x_2)$$

$$= i\, P\left(\frac{\text{"}\phi\text{"}}{x_1,x_2}\right) S_1(y)\,S_2^+(x_1,x_2) + i\,P\left(\frac{x_1}{x_2}\right) S_2(y,x_1)\,S_1^+(x_2) + i\,P\left(\frac{x_1,x_2}{\text{"}\phi\text{"}}\right) S_3(y,x_1,x_2)$$

$$= iS_1(y)\,S_2^+(x_1,x_2) + i\,S_2(y,x_1)\,S_1^+(x_2) + i\,S_2(y,x_2)\,S_1^+(x_1) + i\,S_3(y,x_1,x_2)$$

$$= i^2\mathcal{L}(y)\left(-T(\mathcal{L}(x_1)\mathcal{L}(x_2))\right)^+ - i\,T(\mathcal{L}(y)\mathcal{L}(x_1))(-i)\mathcal{L}(x_2) - i\,T(\mathcal{L}(y)\mathcal{L}(x_2))(-i\mathcal{L}(x_1)) + iS_3(y,x_1,x_2)$$

$$= \mathcal{L}(y)\,\overline{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T(\mathcal{L}(y)\mathcal{L}(x_1))\,\mathcal{L}(x_2) - T(\mathcal{L}(y)\mathcal{L}(x_2))\,\mathcal{L}(x_1) - i^3 S_3(y,x_1,x_2)$$

↑ anti-chronological order (*)

From (21.13) we know that the above expression must vanish if either $y \gtrsim x_1$ or $y \gtrsim x_2$.

(*) $\overline{T}(\mathcal{L}(x)\mathcal{L}(y)) := \theta(y^0-x^0)\,\mathcal{L}(x)\mathcal{L}(y) + \theta(x^0-y^0)\,\mathcal{L}(y)\mathcal{L}(x)$

Since $\mathcal{L}^+(x) = \mathcal{L}(x)$, notice that $\left(T(\mathcal{L}(x)\mathcal{L}(y))\right)^+ = \overline{T}(\mathcal{L}(x)\mathcal{L}(y))$

Disregarding the cases $y \sim x_i$, we have 4 possibilities, corresponding to

(a) $x_1 < y < x_2$
(b) $x_2 < y < x_1$
(c) $x_1 < x_2 < y$
(d) $x_2 < x_1 < y$

From our knowledge of Dyson's series, we expect to obtain

$$S_n(x_1,\dots,x_n) = i^n\, T(\mathcal{L}(x_1)\cdots\mathcal{L}(x_n)). \qquad (*)$$

But, as we shall see, it is only possible to show that $(*)$ is <u>compatible</u> with the 3 conditions we have imposed on the S-matrix (i.e., covariance, unitarity and causality). This is due to the ambiguity in the definition of the T-products. Therefore, an important question we have to ask is: What is the most general form of S_n which is compatible with the 3 conditions?

Let us first write in explicit form the time-ordered product $T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2))$, in order to compare it with $S_3(y,x_1,x_2)$.

We have:

$T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)) =$

$$\begin{aligned}
&= \theta(x_2-x_1)\theta(x_1-y)\,\mathcal{L}(x_2)\mathcal{L}(x_1)\mathcal{L}(y) + \theta(x_1-x_2)\theta(x_2-y)\,\mathcal{L}(x_1)\mathcal{L}(x_2)\mathcal{L}(y)\\
&+ \theta(x_2-y)\theta(y-x_1)\,\mathcal{L}(x_2)\mathcal{L}(y)\mathcal{L}(x_1) + \theta(x_1-y)\theta(y-x_2)\,\mathcal{L}(x_1)\mathcal{L}(y)\mathcal{L}(x_2)\\
&+ \theta(y-x_2)\theta(x_2-x_1)\,\mathcal{L}(y)\mathcal{L}(x_2)\mathcal{L}(x_1) + \theta(y-x_1)\theta(x_1-x_2)\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2).
\end{aligned}$$

Now, what we <u>know</u> from the causality condition is:

$$x_1 < y \ \underline{\text{or}}\ x_2 < y \quad \Rightarrow \quad \begin{aligned} H_2(y;x_1,x_2) = \mathcal{L}(y)\bar{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) &- T(\mathcal{L}(y)\mathcal{L}(x_1))\,\mathcal{L}(x_2)\\ &- T(\mathcal{L}(y)\mathcal{L}(x_2))\,\mathcal{L}(x_1) - i^3 S_3(y,x_1,x_2) \overset{!}{=} 0. \end{aligned}$$

Let us consider the 4 cases (a), (b), (c), (d) described above and see if, in those domains, the expressions for S_3 that follow from $H_2=0$ coincide with the time-ordered product:

(a) $x_1 < y < x_2$: $\quad T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)) = \mathcal{L}(x_2)\mathcal{L}(y)\mathcal{L}(x_1)$

On the other hand, the condition $H_2 = 0$, in this case, leads to:

$$
\begin{aligned}
i^3 S_3(y,x_1,x_2) &= \mathcal{L}(y)\bar{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T(\mathcal{L}(y)\mathcal{L}(x_1))\mathcal{L}(x_2) - T(\mathcal{L}(y)\mathcal{L}(x_2))\mathcal{L}(x_1) \\
&= \mathcal{L}(y)\cancel{\mathcal{L}(x_1)}\mathcal{L}(x_2) - \mathcal{L}(y)\cancel{\mathcal{L}(x_1)}\mathcal{L}(x_2) - \mathcal{L}(x_2)\mathcal{L}(y)\mathcal{L}(x_1) \\
&= -\mathcal{L}(x_2)\mathcal{L}(y)\mathcal{L}(x_1)
\end{aligned}
$$

(b) $x_2 < y < x_1$: $\quad T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)) = \mathcal{L}(x_1)\mathcal{L}(y)\mathcal{L}(x_2)$,

$$
\begin{aligned}
i^3 S_3(y,x_1,x_2) &= \mathcal{L}(y)\bar{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T(\mathcal{L}(y)\mathcal{L}(x_1))\mathcal{L}(x_2) - T(\mathcal{L}(y)\mathcal{L}(x_2))\mathcal{L}(x_1) \\
&= \mathcal{L}(y)\cancel{\mathcal{L}(x_2)}\mathcal{L}(x_1) - \mathcal{L}(x_1)\mathcal{L}(y)\mathcal{L}(x_2) - \mathcal{L}(y)\cancel{\mathcal{L}(x_2)}\mathcal{L}(x_1) \\
&= -\mathcal{L}(x_1)\mathcal{L}(y)\mathcal{L}(x_2)
\end{aligned}
$$

(c) $x_1 < x_2 < y$:

$$T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)) = \mathcal{L}(y)\mathcal{L}(x_2)\mathcal{L}(x_1),$$

$$
\begin{aligned}
i^3 S_3(y,x_1,x_2) &= \mathcal{L}(y)\bar{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T(\mathcal{L}(y)\mathcal{L}(x_1))\mathcal{L}(x_2) - T(\mathcal{L}(y)\mathcal{L}(x_2))\mathcal{L}(x_1) \\
&= \mathcal{L}(y)\cancel{\mathcal{L}(x_1)}\mathcal{L}(x_2) - \mathcal{L}(y)\cancel{\mathcal{L}(x_1)}\mathcal{L}(x_2) - \mathcal{L}(y)\mathcal{L}(x_2)\mathcal{L}(x_1) \\
&= -\mathcal{L}(y)\mathcal{L}(x_2)\mathcal{L}(x_1)
\end{aligned}
$$

(d) $x_2 < x_1 < y$: $\quad T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)) = \mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)$

$$
\begin{aligned}
i^3 S_3(y,x_1,x_2) &= \mathcal{L}(y)\bar{T}(\mathcal{L}(x_1)\mathcal{L}(x_2)) - T(\mathcal{L}(y)\mathcal{L}(x_1))\mathcal{L}(x_2) - T(\mathcal{L}(y)\mathcal{L}(x_2))\mathcal{L}(x_1) \\
&= \mathcal{L}(y)\cancel{\mathcal{L}(x_2)}\mathcal{L}(x_1) - \mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2) - \mathcal{L}(y)\cancel{\mathcal{L}(x_2)}\mathcal{L}(x_1) \\
&= -\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)
\end{aligned}
$$

→ So we have verified that, for those configurations of (y, x_1, x_2) corresponding to cases (a) -(d), the equality

$$S_3(y,x_1,x_2) = i^3\, T(\mathcal{L}(y)\mathcal{L}(x_1)\mathcal{L}(x_2)).$$

holds. The problem is that there is, for the moment being, not much more we can do. We will therefore propose this as an ansatz and check that it complies with the 3 conditions.

For S_3, the causality condition is just what we have done.

What about unitarity? Unitarity requires $(n=3)$

$$S_n(x_1,\dots,x_n) + S_n^+(x_1,\dots,x_n) + \\ + \sum_{1\le k\le n-1} P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right) S_k(x_1,\dots,x_k)\, S_{n-k}^+(x_{k+1},\dots,x_n) = 0$$

Writing this explicitly for $n=3$, we get

$$S_3(x_1,x_2,x_3) + S_3^+(x_1,x_2,x_3) + P\left(\frac{x_1}{x_2,x_3}\right) S_1(x_1)\, S_2^+(x_2,x_3) + P\left(\frac{x_1,x_2}{x_3}\right) S_2(x_1,x_2)\, S_1^+(x_3) =$$

$$= S_3(x_1,x_2,x_3) + S_3^+(x_1,x_2,x_3) + S_1(x_1)\,S_2^+(x_2,x_3) + S_1(x_2)\,S_2^+(x_1,x_3) + \\ S_1(x_3)\,S_2^+(x_1,x_2) + S_2(x_1,x_2)\,S_1^+(x_3) + S_2(x_1,x_3)\,S_1^+(x_2) + S_2(x_2,x_3)\,S_1^+(x_1)$$

$$= -iT(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3) + i\bar{T}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3) - i\mathcal{L}_1\bar{T}(\mathcal{L}_2,\mathcal{L}_3) - i\mathcal{L}_2\bar{T}(\mathcal{L}_1,\mathcal{L}_3) - i\mathcal{L}_3\bar{T}(\mathcal{L}_1,\mathcal{L}_2)$$

$\mathcal{L}_i \equiv \mathcal{L}(x_i)$

$$+ iT(\mathcal{L}_1,\mathcal{L}_2)\,\mathcal{L}_3 + iT(\mathcal{L}_1,\mathcal{L}_3)\,\mathcal{L}_2 + iT(\mathcal{L}_1,\mathcal{L}_3)\,\mathcal{L}_1$$

$= 0$ (exercise!)

For the general case, see Bogoliubov-Shirkov, § 21.5, where a general argument is given to show that the time-ordered exponential satisfies the 3 conditions.

§ 21.6 Analysis of the Arbitrariness of the Functions S_n and the Most General form of $S(g)$

So far we have argued that $S_n(x_1,\dots,x_n) = i^n T(\mathcal{L}(x_1)\cdots\mathcal{L}(x_n))$ is compatible with the 3 requirements we have imposed on the S-matrix. Actually, we have seen that the freedom remaining is produced by the ambiguities produced by the ill-defined time-ordered products along diagonals. We will now explore the nature of this "remaining freedom". First, the unitarity condition (at "level n") involves the linear combination $S_n + S_n^+$. As far as we are only concerned with the causality condition, then, S_n is only determined up to an antihermitean operator, say $i\Lambda_n$ (where $\Lambda_n^+ = \Lambda_n$). In this way we have $(S_n + i\Lambda_n)^+ + (S_n + i\Lambda_n) = S_n^+ - \cancel{i\Lambda_n} + \cancel{i\Lambda_n} + S_n = S_n^+ + S_n$.

Since, by definition, S_n is symmetric in its arguments, Λ_n must also have the same property. Furthermore, as we saw in the example $n=3$, the <u>causality</u> condition completely determines $S_n(x_1, \dots, x_n)$ whenever $x_1 \gtrsim x_j$ (any $j \in \{2, \dots, n\}$). Clearly, in this domain Λ_n must vanish. But then, because of the symmetry in its arguments, the same must be true whenever $x_i \gtrsim x_j$ (for any pair $i, j \in \{1, \dots, n\}$). This in turn implies that $\Lambda_n(x_1, \dots, x_n)$ must vanish whenever $x_i \neq x_j$ (any pair).

The only remaining set for the support of Λ_n is, then, the full diagonal $\Rightarrow$ Λ_n must be a quasi-local operator, of the form

$$\sum p_\alpha(\{\partial_i\})\, \delta(x_1 - x_2) \cdots \delta(x_1 - x_n) :\cdots \varphi_\alpha(x_j) \cdots :$$

p_α: polynomial on $\partial_1, \dots, \partial_n$

field operators in normal form.