La fórmula de Gell-Mann & Low: un ejemplo

Renormalización
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$|0\rangle$: vacío de la teoría libre

$|\Omega\rangle$: vacío de la teoría interactuante

$$\langle\Omega|T(\varphi(x_1)\cdots\varphi(x_n))|\Omega\rangle \overset{!}{=} \frac{\langle 0|T(\varphi(x_1)\cdots\varphi(x_n)\,e^{i\int d^4z\,\mathcal{L}_{int}(z)})|0\rangle}{\langle 0|T(e^{i\int d^4z\,\mathcal{L}_{int}(z)})|0\rangle}$$

campos en el esquema de interacción

(Heisenberg: $\varphi(t,\vec{x}) = e^{itH}\varphi(0,\vec{x})e^{-itH}$)

H: Hamilt. <u>con</u> interacción

campos libres (equiv: esquema de interacción: $\varphi(t,\vec{x}) = e^{itH_0}\varphi(0,\vec{x})e^{-itH_0}$)

Análisis diagramático $\rightarrow$ $\mathcal{L}_{int}(x) = -\lambda\varphi(x)^3$

$$T\left(\exp\left(i\int d^4y\,\mathcal{L}_{int}(y)\right)\right) = \sum_{n\geq 0}\frac{(-i\lambda)^n}{n!}\int d^4y_1\cdots\int d^4y_n\,T(:\varphi(y_1)^3:\cdots:\varphi(y_n)^3:)$$

$$= 1 - i\lambda\int d^4y\,:\varphi(y)^3: \quad + \frac{(-i\lambda)^2}{2}\int d^4y_1\int d^4y_2\,T(:\varphi(y_1)^3::\varphi(y_2)^3:)$$

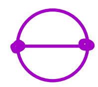
se va en "$\langle 0|\cdot|0\rangle$"

$+\cdots$

$$T\left(:\varphi(y_1)^3::\varphi(y_2)^3:\right) = :\varphi(y_1)^3\varphi(y_2)^3: + 9\,\Delta_F(y_1-y_2):\varphi(y_1)^2\varphi(y_2)^2:$$
$$+ 18\left(\Delta_F(y_1-y_2)\right)^2 :\varphi(y_1)\varphi(y_2):$$
$$+ 6\left(\Delta_F(y_1-y_2)\right)^3$$

$$\Longrightarrow$$

$$\langle 0|T\left(e^{i\int \mathcal{L}_{int}}\right)|0\rangle = 1 - \frac{\lambda^2}{2}\cdot 6 \cdot \int d^4y_1 \int d^4 y_2 \left(\Delta_F(y_1-y_2)\right)^3$$

En diagramas: $1 - 3\lambda^2$ 

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$$\langle 0|T\left(\varphi(x)\varphi(y)e^{i\int\mathcal{L}_{int}}\right)|0\rangle =$$

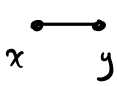
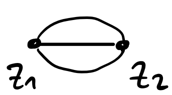
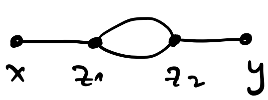
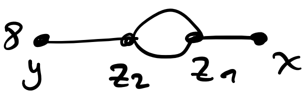
$$= \sum_{n\geq 0}\frac{(-i\lambda)^n}{n!}\int d^4z_1\cdots\int d^4z_n \langle 0|T(\varphi(x)\varphi(y):\varphi(z_1)^3:\cdots:\varphi(z_n)^3:|0\rangle$$

$$= \langle 0|T(\varphi(x)\varphi(y))|0\rangle - i\lambda\int d^4z\, \langle 0|T\cancelto{0}{(\varphi(x)\varphi(y):\varphi(z)^3:)|0\rangle}$$

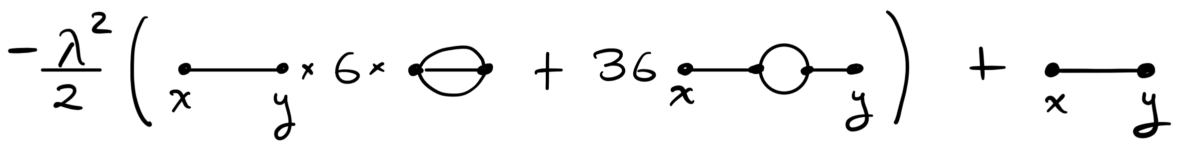
$$+ \frac{(i\lambda)^2}{2}\int d^4z_1\int d^4z_2 \underbrace{\langle 0|T(\varphi(x)\varphi(y):\varphi(z_1)^3::\varphi(z_2)^3:)|0\rangle}_{(*)}$$

$$+ \cdots$$

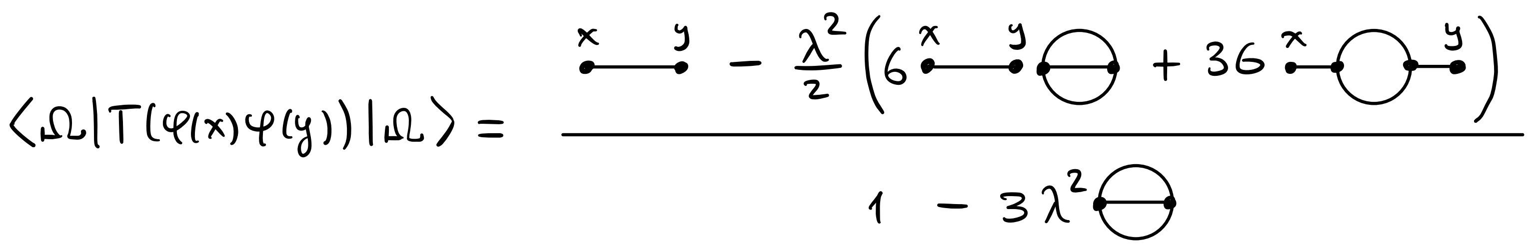
De manera simbólica, para el término $(*)$, el teorema de Wick nos da: $\langle 0|T(\varphi(x)\varphi(y):\varphi(z_1)^3::\varphi(z_2)^3:)|0\rangle =$

$=$  $\times 6 \times$  $+ 18$  $+ 18$ 

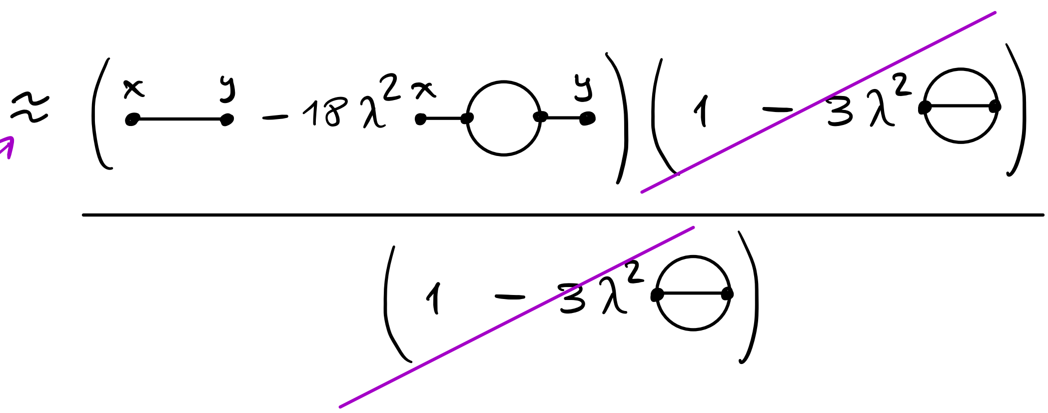
... el resultado, luego de integrar sobre z_1 y z_2, puede ser representado diagramáticamente de la siguiente forma:



De esta forma vemos que la contribución del denominador se cancela con algo que viene del numerador:



a orden λ^2...



En general $\longrightarrow$

V_i : diagrama de vacío

$$\langle \Omega | T(\varphi(x_1)\cdots\varphi(x_n)) | \Omega \rangle = \frac{\left[\begin{array}{l}\text{suma de diagramas} \\ \text{con n-patas}\end{array}\right] \times \cancel{e^{\sum_i V_i}}}{\cancel{e^{\sum_i V_i}}}$$

Volvamos al cálculo, de forma explícita:

Propagador: $\langle \Omega | T(\varphi(x)\varphi(y)) | \Omega \rangle$

$$\langle 0 | T(\varphi(x)\varphi(y)\, e^{i\int d^4z\, \mathcal{L}_{int}(z)} \,|0\rangle$$

$$= \Delta_F(x-y) - i\lambda\, \langle 0 | T(\varphi(x)\varphi(y) \int d^4z :\varphi(z)^3:)\,|0\rangle$$

$$+ \frac{(-i\lambda)^2}{2!} \langle 0 | T(\varphi(x)\varphi(y) \int d^4z_1 \int d^4z_2 :\varphi(z_1)^3::\varphi(z_2)^3: |0\rangle$$

$$= \Delta_F(x-y) + \frac{(-i\lambda)^2}{2!}\cdot 36 \int d^4z_1 \int d^4z_2\, \Delta_F(x-z_1)\Delta_F(y-z_2)\, \underbrace{(\Delta_F^2)_{fin}(z_1-z_2)}_{=\ \frac{1}{(2\pi)^2}\int d^4k\, e^{-i(z_1-z_2)\cdot k}\, I_{fin}(k^2,\mu)},$$

(asumiendo que hemos introducido contratérminos)

$$= \Delta_F(x-y) + (-i\lambda)^2 \frac{\cdot 18}{(2\pi)^2} \int d^4z_1 \int d^4z_2 \int d^4k\, e^{-i(z_1-z_2)\cdot k}\, \Delta_F(x-z_1)\Delta_F(y-z_2)\, I_{fin}(k^2,\mu)$$

$\delta(k-p_1)\,\delta(k+p_2)$ $\qquad$ $k = p_1 = -p_2$

$$= \Delta_F(x-y) + -\frac{(-i\lambda)^2\cdot 18}{(2\pi)^{10}} \int d^4z_1 \int d^4z_2 \int d^4k\, e^{-i(z_1-z_2)\cdot k} \int d^4p_1 \int d^4p_2\, \frac{e^{-ip_1\cdot(x-z_1)}\, e^{-ip_2(y-z_2)}}{(p_1^2-m^2)(p_2^2-m^2)}\, I_{fin}(k^2,\mu)$$

$$= \frac{i}{(2\pi)^4}\int d^4k\, \frac{e^{-ik\cdot(x-y)}}{k^2-m^2} + \frac{18\lambda^2}{(2\pi)^2}\int d^4k\, \frac{e^{-ik(x-y)}}{(k^2-m^2)^2}\, I_{fin}(k^2,\mu)$$

$$= \int d^4k\, e^{-ik\cdot(x-y)} \left(\frac{i}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon} + \frac{i}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon} \left[\underbrace{-\frac{9\lambda^2}{(2\pi)^2}(2\pi)^8\, I_{fin}(k^2,\mu)}_{=:\ i(2\pi)^4\, \Sigma(k^2,\mu)} \right] \frac{i}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon} \right)$$

$$= \int d^4k \, e^{-ik\cdot(x-y)} \left(\frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} + \frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \left[i(2\pi)^4 \sum(k^2,\mu) \right] \frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} + \dots \right)$$

$$= \int d^4k \, e^{-ik\cdot(x-y)} \left(\frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \right) \Bigg[1 + \left[i(2\pi)^4 \sum(k^2,\mu) \right] \frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} + \left(\left[i(2\pi)^4 \sum(k^2,\mu) \right] \frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \right)^2 + \dots \Bigg]$$

$$= \int d^4k \, e^{-ik\cdot(x-y)} \left(\frac{i}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \right) \frac{1}{1 - \left[i \sum(k^2,\mu) \right] \frac{i}{k^2-m^2+i\varepsilon}}$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2-m^2+\sum(k^2,\mu)+i\varepsilon}$$

$\longrightarrow$

$$\langle \Omega | T(\varphi(x)\varphi(y)) | \Omega \rangle = \frac{i}{(2\pi)^4} \int d^4k \, \frac{e^{-ik\cdot(x-y)}}{k^2-m^2+\sum(k^2,\mu)+i\varepsilon}$$