Primer Encuentro de Geometría Diferencial en Física

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Andrés Reyes.

1<u>er</u> Encuentro de Geometría Diferencial en Física

Física en bajas dimensiones
Marzo 3, 4, 5 1994

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Presentación

A lo largo de la historia de la ciencia, la física y la matemática han mantenido una relación muy estrecha; en algunos casos el desarrollo de algún concepto físico impulsa alguna disciplina matemática; en otros el desarrollo formal, o el entendimiento preciso de algún concepto físico, requiere del uso de lenguaje matemático desarrollado inicialmente con propósitos diferentes. Por eso resulta importante la búsqueda de espacios, en los cuales puedan ser discutidas las nuevas formas de interacción entre estas disciplinas. De manera concreta, el empleo de procedimientos y conceptos de la geometría diferencial encuentra día a día más posibilidades en ramas de la física apartentemente diferentes como lo constituyen la teoría cuántica de campos y la física de la materia condensada, las cuales parecen recuperar algo de su identidad original a través de problemas que se pueden reducir de alguna forma a problemas de indole geométrico.
Ofrecemos por lo tanto en estas memorias un primer ejemplo, que muestra las posibilidades de intercambio entre la Física y la Matemática para entender diversas inquietudes que son pertinentes en el desarrollo interno de las dos disciplinas.

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SOME NOTES ON GEOMETRY AND QUANTIZATION

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Gauge theory of particle physics rests on the observation that the basic physical concepts (fields and particles) cannot, in general, be described canonically by elementary mathematics —the mathematics of functions and calculus. Rather calculus is the 'vacuum' or 'least energy' state of mathematics, where the mathematics, and hence the physics, is essentially 'trivial'. It postulates, nevertheless, that in small patches of space-time the mathematics is trivial, though in general highly non-canonically —that is, it is trivial but one must choose a frame of reference (coordinates) in order to achieve that triviality. Such a choice is called a 'gauge' and changing the frame of reference is called a 'gauge transformation'. The crucial requirement of a gauge theory is that it requires the physics to be invariant under local gauge transformations (ie, ones that vary between different points). That means something is unmeasurable and that implies a conserved quantity, for example, 'charge' in electromagnetism or 'colour' in SU(2)-Yang-Mills theories. The relation between gauge symmetries and conservation laws is made precise by Noether's Theorem.

The global non-triviality of a system is encoded in the way local descriptions are joined together, and that is topology. Topology tells us about intrinsic global properties of a space, essentially how many holes there are in it. Perhaps the best known example of that is the Euler number $\chi(\Sigma)$ of a compact surface Σ. One finds that

$$\chi(\Sigma) = 2 - 2g$$

where g =number of holes in Σ, so $\chi(\text{sphere}) = 2$, $\chi(\text{torus}) = 0$. Moreover, two surfaces Σ_1 and Σ_2 are isomorphic if and only if $\chi(\Sigma_1) = \chi(\Sigma_2)$. So the Euler number is a topological invariant and gives a 'functor'

$$\begin{aligned} \chi : \{\text{Space of surfaces}\} &\longrightarrow \mathbb{Z} \\ \Sigma &\longmapsto \chi(\Sigma). \end{aligned}$$

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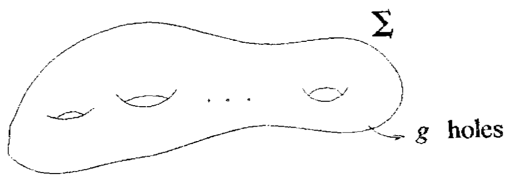


Because the dynamics of physical systems determine associated topological phase and configuration spaces, we see that fundamental physical characteristics may be reflected in the topology. Different fundamental particles will have topologically distinct dynamical spaces.

However, gauge theory is not topology. Gauge theory mathematically describes classical mechanics and field theory, and that is differential geometry. Differential geometry is essentially concerned with the local properties of space determined by its differential and unitary (metric) structure (e. g. curvature, lengths of curves, volume). The enchanting aspect is that we find that differential geometry determines or tell us about the topology of a space. Perhaps the prototypical example of this is the Gauss-Bonnet Theorem which states that

$$\chi(\Sigma_g) = \frac{1}{2\pi}\int_{\Sigma_g} K_{\Sigma_g}\, du\, dv$$

where K_{Σ_g} denotes the Gaussian curvature of the surface Σ_g. This presages much deeper relations between differential geometry and topology, most notably the Atiyah-Singer index theorem.

Physically, the movement from the continous phenomena described by differential geometry to the discrete phenomena of topology is best seen as a movement from classical (continous energy) physics to quantum (discrete energy) physics. Indeed the numbers $\chi(\Sigma_g)$ are examples of what physicists call 'topological quantum numbers'. Thus gauge theory is used to formulate classical field theory which one then seeks to quantize in some natural way to get the corresponding quantum theory. It should be noted that this may be a purely formal procedure, e. g. for the weak and strong force, because the theories may not exist as classical field theories, or perhaps an impossible procedure.

These notes represent the content of three rapid introductory lectures to differential geometry. Consequently, they do not attempt to be at all comprehensive. The first part does treat the construction of the tangent bundle in some detail since the tangent bundle is the prototypical example of a vector bundle. I believe one can avoid reading many arduous text books by understanding one or two basic examples thoroughly, and so I hope this account may assist in that process. The second part presents a quick review of some of the ideas concerning vector bundles in general. Finally,

the third section introduces connections very rapidly and presents an outline of the construction of a $0+1$-dimensional topological quantum field theory.

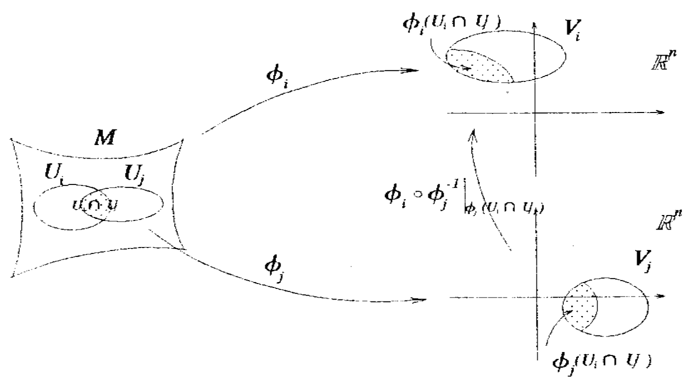
1. The Tangent Bundle

1.1 Manifolds.

In Newtonian mechanics one considers the universe to be 4-dimensional, corresponding to space and time, and to be 'flat'. One should not therefore conclude that the universe is a copy of $\mathbb{R}^4$. Before Galileo that was (not unreasonably) the assumption in the geocentric model of the universe with the Earth at its centre as a preferred point, the origin. However, it is now generally accepted that the Earth is just a point and one cannot say there is a natural centre or origin. In other words, space-time is flat affine space A^4 —Of course, this is the Newtonian picture; it could be that the universe has a hugely complicated global geometry. A^4 is distinguished from $\mathbb{R}^4$ in that there is no fixed origin. The group $\mathbb{R}^4$ acts on A^4 by parallel displacements ($v \mapsto v+a$, $v \in A^4$, $a \in \mathbb{R}^4$), so the sum of true points is not defined, but their difference is defined and is a vector in $\mathbb{R}^4$. It is, nevertheless, often useful to choose some point in A^4 to be the origin so that we can give different points coordinates relative to the choice of origin and also talk about lengths by introducing a Euclidean structure $\|\cdot\|$. Thus although A^4 exists intrinsically we find it convenient to identify it with $\mathbb{R}^4$ to be able to work in 'local' (though in this case 'global') coordinates. Manifolds are the generalization of this idea to curved space. To give the precise definition first let us recall that a topological space is a set X with a topology τ. A topology τ on X is a family of subsets $\tau = \{\, U_\alpha : \alpha \in I \,\}$ of X, called neighbourhoods, or open sets, such that the empty set $\emptyset$ and X belong to τ, the union of any number of open sets belongs to τ, the intersection of any finite number of open sets belongs to τ. Once a set is endowed with a topology one can talk about continuity, about continuous functions on X. In the case of $\mathbb{R}^n$ that is equivalent to the usual ϵ-δ definition of continuity when $\mathbb{R}^n$ is given the metric topology (generated by) $\tau_{\text{metric}} = \{\, U_\alpha(y) : \alpha \in \mathbb{R}^+,\, y \in \mathbb{R}^n \,\}$, $U_\alpha(y) = \{\, x \in \mathbb{R}^n : \|x-y\| < \alpha \,\}$, where $\|\cdot\|$ denotes the Euclidean norm. To be more precise, if (X,τ_1) and (Y,τ_2) are topological spaces then a mapping $f : X \to Y$ is said to be continuous if $f^{-1}(U) \in \tau_1$ for every $U \in \tau_2$, ie the inverse image of any open subset is open.

The basic notion of topological equivalence is homeomorphism. We say, in the above situation, that X and Y are homeomorphic if there is a bijective mapping $f : X \to Y$ which is continous and has continuous inverse f^{-1}; f is then said to be a homeomorphism.

We can now formally define an *n-dimensional manifold* M to be a Hausdorff topological space such that each point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. Thus a manifold is a topological space that locally "looks like" $\mathbb{R}^n$. 'Hausdorff' means that for any two distinct points $x_1, x_2 \in M$ there exist disjoint open sets U_1, U_2 in the topology of M, containing x_1 and x_2 respectively, ie $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$; this is included to avoid 'pathological' cases, so M really does look like $\mathbb{R}^n$ locally. Notice that to show M is a manifold we have only to find a covering of M by open sets $\{\, U_i : i \in \Lambda \,\}$ (so each $U_i \in \tau$) and a corresponding homeomorphism $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ (V_i open), ie we do not have to find such a homeomorphism for every open set in the topology. We call the pair (U_i, ϕ_i) a chart for M and the collection of charts $\mathcal{A} = \{ (U_i, \phi_i) : i \in \Lambda \}$ an atlas for M. This works on precisely the same principle as a geographical atlas; each region of the world is covered by a page (ie a chart) in the atlas describing the geographical features of the region. To move from one region to another one moves between different pages (charts) and there are precise instructions for how to do this*. These transitions are achieved in the case of manifolds by transition functions which are homeomorphisms that move from one chart to another. To be precise we have the following diagram of maps:



that is, the following diagram commutes

$$\begin{array}{ccc} U_i \cap U_j & \xrightarrow{\ \phi_i\ } & \phi_i(U_i \cap U_j) \\ \text{id}\downarrow & & \uparrow \phi_i \circ \phi_j{}^{-1} \\ U_i \cap U_j & \xrightarrow[\ \phi_j\ ]{} & \phi_j(U_i \cap U_j) \end{array}$$

(where, of course, ϕ_i means the restriction of $\phi_i : U_i \to V_i$ to $U_i \cap U_j \subset U_i$).

The homeomorphisms $\phi_{ij} = \phi_i \circ \phi_j{}^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are the transition functions of the atlas.

To each chart (U_i, ϕ_i) on M we have an associated system of local coordinates $x = (x_1, \ldots, x_n)$ defined by

$$x_k = u_k \circ \phi_i : U_i \subset M \to \mathbb{R}^1$$

where $u_k : \mathbb{R}^n \to \mathbb{R}^1$, $u_k(a_1, \ldots, a_n) = a_k$, is the k^{th} canonical projection (or coordinate) function on $\mathbb{R}^n$, ie its picks outthe k^{th} component of the vector in $\mathbb{R}^n$. So we give X coordinates in U_i by borrowing them from $\mathbb{R}^n$. However, if $x \in M$ lies in the

*Indeed many of our ideas in geometry are a direct result of navigation of the Earth, in particular the geometric description of a sphere is really the same as cartography (mapping) of planet Earth.

overlap of two charts (U_i, ϕ_i), (U_j, ϕ_j) then we have two different coordinates for x, namely

$$x \longleftrightarrow (x_1, \ldots, x_n) \quad \text{where } x_k = u_k \circ \phi_i : U_i \cap U_j \to \mathbb{R}^1$$

and

$$x \longleftrightarrow (y_1, \ldots, y_n) \quad \text{where } y_k = u_k \circ \phi_j : U_i \cap U_j \to \mathbb{R}^1.$$

But that is fine because we know precisely how they are related, the transition function ϕ_{ij} takes one to the other; $\phi_{ij} : (y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$.

Now one of our basic aims is to do calculus on manifolds, so that means we need to know how to differentiate functions on M. Well, again we borrow the idea from $\mathbb{R}^n$ using the coordinate charts —because we only know what 'derivative' means in Euclidean space; we build using what we already know, not what we don't. First, recall that a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^1$, where U is open, is differentiable of class C^k on U for $k \in \mathbb{N} \cup \{0\}$, if the partial derivates $\frac{\partial^r f}{\partial u_i{}^r}$ exist and are continuous on U for $r \leq k$. A map $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is C^k if each of its component functions $f_i = u_i \circ f$, $i = 1, \ldots, m$ are C^k. If f is C^k for all $k \geq 0$ then we say that f is C^∞ or 'smooth'. We shall only be concerned with C^∞ maps.

To talk about smooth functions on a manifold M we have to augment the topological structure with a differentiable structure (That is, rather than talk about homeomorphisms we must talk about their smooth counterparts which are called diffeomorphisms). The basic refinement we need is to endow M with a C^∞-atlas; that is, an atlas $\mathcal{A} = \{ (U_i, \phi_i) : i \in \Lambda \}$ for M such that the transition functions $\phi_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are C^∞. In fact, that means they are diffeomorphisms; they are smooth homeomorphisms with smooth inverses. The C^∞-atlas is said to define a differentiable structure on M if it is maximal —that means, if (U, ϕ) is a chart on M such that $\phi \circ \phi_i{}^{-1}$ and $\phi_i \circ \phi^{-1}$ are C^∞ for all $i \in \Lambda$, then $(U, \phi) \in \mathcal{A}$. In fact, to show a manifold has a differentiable structure it is enough to find just one C^∞-atlas $\mathcal{A}$, because there is always a unique differentiable structure $\mathcal{F}$ containing $\mathcal{A}$, namely

$$\mathcal{F} = \{ (U, \phi) : U \in \tau, \phi \circ \phi_i{}^{-1} \text{ and } \phi_i \circ \phi^{-1} \text{ are } C^\infty \text{ for all } i \in \Lambda^* \}.$$

Given a differentiable structure, M is then said to be a differentiable (or smooth) manifold of dimension n.

We can now define what we mean by a C^∞ function $f : M \to \mathbb{R}^1$. Let $\mathcal{A} = \{ (U_i, \phi_i) : i \in \Lambda \}$ be a C^∞-atlas on M. Then $f : M \to \mathbb{R}^1$ is differentiable or C^∞ or smooth if $f \circ \phi_i{}^{-1} : V_i \subset \mathbb{R}^n \to \mathbb{R}^1$ is C^∞ for each coordinate patch, ie for each $i \in \Lambda$ (Note that $f \circ \phi_i{}^{-1}$ is a map between Euclidean spaces where C^∞ is well defined). We denote the space of all C^∞ functions on M by $C^\infty(M)$.

More generally, if M and N are differentiable manifolds and $f : M \to N$ is a map between them then f is C^∞ if for each C^∞ coordinate chart (U, ϕ) on M and each

*ie $(U_i, \phi_i) \in \mathcal{A}$.

C^∞ coordinate chart (W, ψ) on N (ie lying in C^∞ atlases) the function

$$\psi \circ f \circ \phi^{-1} : V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad \phi(U) = V \subset \mathbb{R}^m$$
$$m = \dim M, \qquad n = \dim N$$
$$\begin{pmatrix} V & \longrightarrow & \mathbb{R}^n \\ \downarrow \phi^{-1} & & \uparrow \psi \\ M & \xrightarrow{f} & N \end{pmatrix}$$

is C^∞. We denote the space of all C^∞ maps $M \to N$ by $C^\infty(M; N)$. In particular, if $f^{-1} : N \to M$ is also C^∞ and f is a homeomorphism then f is called a diffeomorphism between M and N. This is the basic notion of equivalence in differential topology.

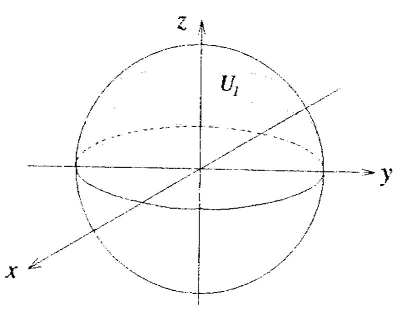
Now lets see some simple examples of manifolds.

Example 1.1. $\mathbb{R}^n$ with de canonical atlas $\mathcal{A} = \{, (\mathbb{R}^n, \mathrm{id})\}$ (id =identity map) consisting of a single chart is trivially a C^∞ manifold. More generally, any vector space V (finite-dimensional) has a natural C^∞-manifold structure. If $\{e_i\}$ is a basis for V, then the dual basis $\{e_i{}^*\}$ defines a global coordinate system on V. That defines a C^∞-structure on V which is independent of the choice of the basis (because different bases have C^∞ transitions on the overlaps).

Example 1.2. The n-sphere $S^n \subset \mathbb{R}^{n+1}$ defined by

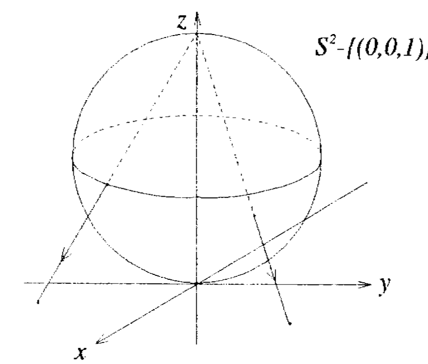
$$S^n = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \},$$

is naturally a manifold. Probably the easiest way to define a C^∞-atlas for S^n is by projection onto the hyperplanes $x_i = 0$. For example, for $n = 2$



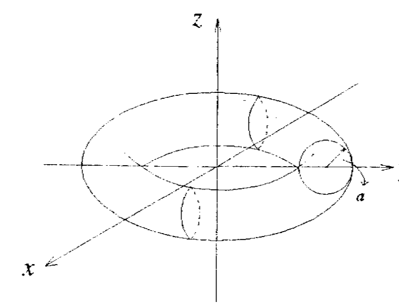
project upper-half sphere onto xy-plane.
So $U_1 = \{ (x, y, z) \in S^2 : z > 0 \}$,
$\phi_1 : U_1 \to \mathbb{R}^2$, $\phi_1(x, y, z) = (x, y)$.
This atlas for S^2 requires 6 charts.

We can though be more economical in our choice of atlas by using first two charts via the stereographic projection atlas. One takes the charts $S^n \setminus \{(0, \ldots, 0, 1)\}$ and $S^n \setminus \{(1, 0, \ldots, 0)\}$ and the projects via lines emanating from the north and south poles. For example, for $n = 2$



The details are left as an exercise for the reader.

Example 1.3. The torus



can be two charts of the form

$$U = (0, 2\pi) \times (0, 2\pi)$$
$$\phi(\vartheta, \varphi) = \big((a + r\sin\vartheta)\cos\varphi, (a + r\cos\vartheta)\sin\varphi, r\sin\vartheta\big).$$

This is an example of a surface of revolution —obtained in this example by rotating a copy of S^1 around the z-axis.

Example 1.4. *Product manifolds.*
If M and N are differentiable (C^∞) manifolds of dimension m and n, then the cartesian product $M \times N$ has a natural C^∞ manifold structure with atlas $\mathcal{A}_{M\times N} = \{ (U_M \times U_N, \phi_M \times \phi_N) : (U_M, \phi_M) \in \mathcal{A}_M, (U_N, \phi_N) \in \mathcal{A}_N \}$ where $\mathcal{A}_M$ and $\mathcal{A}_N$ are, respectively, C^∞ atlases on M and N.

In particular this gives another construction of the C^∞-manifold structure for a torus $T^2 = S^1 \times S^1$ —and, more generally, for an n-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$.

Example 1.5. Yet another construction of the (n-)torus is given by taking a quotient of $\mathbb{R}^{2n}$. Let $\omega_1, \ldots, \omega_{2n}$ be vectors in $\mathbb{R}^{2n}$ linearly independent over $\mathbb{Z}$, and let Γ be the lattice defined by

$$\Gamma = \{ n_1\omega_1 + \cdots + n_{2n}\omega_{2n} : n_i \in \mathbb{Z} \}.$$

So $v_1, v_2 \in \mathbb{R}^{2n}$ are equivalent $\mod (\Gamma)$ iff $v_1 - v_2 \in \Gamma$. This defines an equivalence relation and the set of equivalence classes is the quotient space torus

$$T^n = \mathbb{R}^{2n}/\Gamma.$$

T^n is given the quotient topology, so $U \subset \mathbb{R}^{2n}/\Gamma$ is open if and only if $\pi^{-1}(U) \subset \mathbb{R}^{2n}$ is open, where $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\Gamma$ is the quotient map. In particular, because $\mathbb{R}^{2n}$ is connected then so is $\mathbb{R}^{2n}/\Gamma$. Further because π is surjective on compact subsets of $\mathbb{R}^{2n}$ then $\mathbb{R}^{2n}/\Gamma$ is compact (—these are both consequences of the quotient topology). To see the manifold structure, let $U \subset \mathbb{R}^{2n}$ be an open set such that no two points are equivalent $\mod \Gamma$ —so $U/\Gamma \cong U$. Then $\pi(U)$ is open and $\pi : U \to U/\Gamma$ is a homeomorphism. A chart on T^n is then defined by $(\pi(U), \pi^{-1})$. Covering all of T^n by such charts it's easy to see this defines a C^∞ atlas —the transition functions are constant and hence C^∞.

A diffeomorphism of T^n as defined here with T^n as defined in Example 1.4 is given by

$$(\lambda, \mu, \ldots, \nu) \mapsto (e^{2\pi i\lambda}, e^{2\pi i\mu}, \ldots, e^{2\pi i\nu}).$$

Example 1.6. *Real projective space.*
This is defined as

$$\begin{aligned}\mathbb{R}P^n &= \{ \text{one dimensional subspaces of } \mathbb{R}^{n+1} \} \\ &= \mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}.\end{aligned}$$

To see why this has a natural C^∞ manifold structure we proceed in a similar way to Example 1.5. Define the quotient map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$$

by

$$\pi(x) = \text{1-dimensional subspace spanned by } x = (x_0, \ldots, x_n).$$

Using π we give $\mathbb{R}P^n$ the quotient topology so that $U \subset \mathbb{R}P^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$ in the metric topology. So π, by definition, is continuous and $\mathbb{R}P^n$ is Hausdorff. Because $\pi_{|S^n} : S^n \to \mathbb{R}P^n$ is continuous and surjective and S^n is compact then $\pi(S^n) = \mathbb{R}P^n$ is compact. To prove that $\mathbb{R}P^n$ has a manifold structure we define homogeneous coordinates on $\mathbb{R}P^n$; if $\omega \in \mathbb{R}P^n$ then $\omega = \pi(x) = \pi((x_0, \ldots, x_n))$ for some $x \in \mathbb{R}^{n+1}$, (non-unique, of course) then $(x_0, \ldots, x_n)$ are said to be homogeneous coordinates of $\pi(x) = \omega = [x_0, \ldots, x_n]$. If $(x'_0, \ldots, x'_n)$ are another

set of homogeneous coordinates then $x'_i = \lambda x_i$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Clearly one has $\pi(x) = \pi(\mu x)$ for any $\mu \in \mathbb{R} \setminus \{0\}$. Now define $(n+1)$-charts for $\mathbb{R}P^n$ by

$$U_\alpha = \{ \omega \in \mathbb{R}P^n : \omega = [x_0, \ldots, x_n] \text{ and } x_\alpha \neq 0 \} \quad \alpha = 0, \ldots, n$$

and

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n,$$

$$\phi_\alpha([x_0, \ldots, x_n]) = \left(\frac{x_0}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_n}{x_\alpha} \right).$$

Clearly ϕ_α is a homeomorphism and $\phi_\alpha \circ \phi_\beta{}^{-1}$ is a diffeomorphism for $\alpha, \beta = 0, \ldots, n$, and that defines $\mathbb{R}P^n$ as a C^∞-manifold. An instructive exercise I leave to the reader is to prove that the map

$$\mathbb{R}P^1 \longrightarrow S^1 \qquad [x, y] \longmapsto \left(\frac{x}{(x^2+y^2)^{1/2}}, \frac{y}{(x^2+y^2)^{1/2}} \right)$$

is a diffeomorphism.

Example 1.7. *Grassmanians.*
Real projective space of Example 1.6 is a special case of a Grassmanian manifold. We define

$$\mathrm{Gr}_k(\mathbb{R}^n) = \{ k\text{-dimensional subspaces of } \mathbb{R}^n \}.$$

Thus each point of $\mathrm{Gr}_k(\mathbb{R}^n)$ parametrizes a k-dimensional subspace of $\mathbb{R}^n$. Clearly

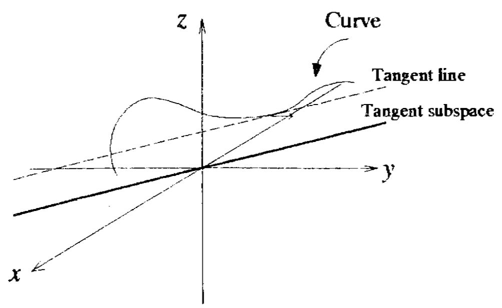
$$\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{R}P^n.$$

There is a Grassmanian for each $k = 1, \ldots, n$ and $\mathrm{Gr}_k(\mathbb{R}^n)$ is a C^∞ manifold of dimension $k(n-k)$. The C^∞ structure is defined in a similar way as for $\mathbb{R}P^n$, but I shall omit the details. Grassmanians play a crucial role in vector bundle theory, representation of compact Lie groups, and in (geometric) quantization.

1.2 Tangent Bundles and Vector Fields.

Everything we do here is answering the question: how do we talk about vector fields on curved space, ie on manifolds? We know what a vector field is on $\mathbb{R}^n$, and, in answering this question, one eventually concludes that when it is not possible to describe a vector field on a manifold in terms of a vector field on $\mathbb{R}^n$, then that precisely reflects the topology of the manifold (after all topology is all about how manifolds differ from $\mathbb{R}^n$).

Recall that by a vector field on $\mathbb{R}^n$ one essentially means a map $\mathbb{R}^n \to \mathbb{R}^n$ that associates to each point of $\mathbb{R}^n$ a vector in $\mathbb{R}^n$. Moreover, each such vector may be realised as the derivate to some curve in $\mathbb{R}^n$ passing through that point. One often



refers to such a derivate as a tangent vector to the curve and which hence defines a tangent line to the curve.

If we translate that line in a parallel way so that it passes through the origin of $\mathbb{R}^n$ then it defines a 1-dimensional subspace of $\mathbb{R}^n$. By considering curves in different directions one sees that there is actually an n-dimensional vector space of such derivatives or tangent vectors associated to each point of $\mathbb{R}^n$. But now it is getting a bit confusing because I referred to the tangent space to a curve as a subspace of $\mathbb{R}^n$, so presumably the space of all tangent vectors is a subspace of $\mathbb{R}^n$, but because that space (of tangent vectors) is equal to $\mathbb{R}^n$ then the space of all tangent vectors at a point of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. Well, of course, the point is that one must distinguish between the vector space defined by tangent vectors to a point of $\mathbb{R}^n$ from $\mathbb{R}^n$ itself. It is distinct from, though isomorphic to, $\mathbb{R}^n$. To do that we make the following observation. Let $x \in \mathbb{R}^n$ and $\sigma : \mathbb{R}^1 \to \mathbb{R}^n$ be a curve with $\sigma(0) = x$, $\sigma'(0) = v = (v_1, \ldots, v_n)$. Then given any smooth function $f \in C^\infty(\mathbb{R}^n)$ one has a directional derivate

$$\frac{d}{dt} f(\sigma(t))\Big|_{t=0} = \nabla f_x \cdot v = v_1 \frac{\partial f}{\partial u_1}\Big|_x + \cdots + v_n \frac{\partial f}{\partial u_n}\Big|_x \tag{2.1}$$

where ∇ is the usual gradient operator. So we see that one may regard tangents to curves as acting on differentiable functions on $\mathbb{R}^n$ in the sense that

$$\begin{aligned} v = \sigma'(0) &: C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}^1 \\ v(f)(x) &= v_1 \frac{\partial f}{\partial u_1}\Big|_x + \cdots + v_n \frac{\partial f}{\partial u_n}\Big|_x . \end{aligned} \tag{2.2}$$

That is, we identify the vector

$$v = v_1 e_1 + \cdots + v_n e_n \in \mathbb{R}^n$$

with the first-order differential operator

$$v = v_1 \frac{\partial}{\partial u_1}\Big|_x + \cdots + v_n \frac{\partial}{\partial u_n}\Big|_x : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}^1.$$

That is we identify the vectors

$$e_i \longleftrightarrow \frac{\partial}{\partial u_i}\Big|_x .$$

If you like, this is a 'quantization' of the elementary idea of tangent vector as a derivative to a curve. Clearly for each such derivative there is a map of the form (2.2) and the two interpretations are equivalent. In this spirit there we may define the tangent space T_x to $\mathbb{R}^n$ at $x \in \mathbb{R}^n$ as the vector space over $\mathbb{R}$ spanned by the n-vectors

$$\left\{ \frac{\partial}{\partial u_1}\Big|_x , \ldots , \frac{\partial}{\partial u_n}\Big|_x \right\} .$$

Well, that is certainly a step forward, we now see that a tangent vector is something that defines the derivative to a given function in a direction defined by the tangent vector —ie a directional derivative— and so it is clear that T_x is an n-dimensional vector space associated to $x \in \mathbb{R}^n$ but distinct from $\mathbb{R}^n$. However, we can do better. We note that each $\frac{\partial}{\partial u_i}\Big|_x$ and hence the vector v in (2.1) satisfies two basic properties:

(P1) $\quad v(f + \lambda g)(x) = v(f)(x) + \lambda v(g)(x) \quad$ (linear)

(P2) $\quad v(f \cdot g)(x) = f(x) v(g)(x) + v(f)(x) g(x) \quad$ (derivation).

These are, in fact, characterizing properties. More precisely, we can give the space of linear derivations (ie operators v on functions satisfying (P1) and (P2)) a natural vector space structure by defining

$$(v_1 + v_2)(f) = v_1(f) + v_2(f) \tag{2.3}$$

$$(\lambda v)(f) = \lambda(v(f)). \tag{2.4}$$

We denote this vector space of linear derivations at x by $T_x\mathbb{R}^n$. One has the following crucial result:

Theorem. *There is a canonical isomorphism*

$$T_x\mathbb{R}^n \cong T_x .$$

I shall not prove that result here since we will not need it, other than as a guide for how to proceed. I should also mention 2 points I have glossed over:

(1) We should really think of linear derivations as defined on 'germs' of functions. These are equivalence classes of functions which agree on a neighbourhood of $x \in \mathbb{R}^n$. That, however, while technically accurate is not helpful, and for all relevant purposes it is enough to consider derivations just on functions.
(2) It is enough to consider derivations as defined on functions which are C^∞ in a neighbourhood of the point $x \in \mathbb{R}^n$ in question. But again that is a refinement that it is irrelevant for our purposes here.

This identification asserted by the theorem is a crucial observation. It is not just mathematicians playing abstract matheatical games, but underlies the essential idea of how to do calculus on manifolds and eventually gauge theory. The point is this, we see that the tangent space to $x \in \mathbb{R}^n$ can be characterised in a purely intrinsic or global way in terms of derivations. This characterization is intrinsic\global in the sense that no additional choices are necessary to state it, no coordinates are needed. The characterization in terms of the familiar derivative operators $\left\{ \frac{\partial}{\partial u_i}\Big|_x \right\}$ involves a canonical basis for $T_x\mathbb{R}^n$. That is, each linear derivation can be canonically identified as a linear combination of the vectors $\frac{\partial}{\partial u_i}\Big|_x$ —once we select this basis. That is important for telling us how to define tangent spaces on abstract manifolds, because there we do not usually have a global coordinate system and so the only possible type of definition is an intrinsic one. Once we have the intrinsic definition we can investigate what tangent vectors look like locally relative to some local coordinate system. Anyway, let us make the following definition.

Definition [2.1]. The tangent space $T_x\mathbb{R}^n$ to $\mathbb{R}^n$ at x is the space of linear derivations at x (ie (P1), (P2)) with the vector space structure defined by (2.3) and (2.4).

(†) Once we choose the canonical coordinate system $(u_1, \dots, u_n)$ on $\mathbb{R}^n$ an element $v \in T_x\mathbb{R}^n$ can be written as
$$v = \alpha_1 \frac{\partial}{\partial u_1}\Big|_x + \cdots + \alpha_n \frac{\partial}{\partial u_n}\Big|_x,$$
for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}^1$.

Because definition [2.1] is intrinsic we can copy it straight away to an arbitrary smooth n-manifold M.

Definition [2.2]. The tangent space T_mM to M at $m \in M$ is the space of linear derivations defined on C^∞ functions on M (or near m) with the vector space structure defined by (2.5) and (2.4).

So that is fine, but at this point we cannot repeat the statement (†) above for M. Our next task is to investigate the analogue of (†) for M.

Our first observation is the following.

Proposition [2.3]. *Let M and N be C^∞-manifolds and let $\psi : M \to N$ be a C^∞-map. Then there is a natural linear map*
$$d\psi : T_mM \longrightarrow T_{\psi(m)}N$$
defined by
$$d\psi(v)(f)(\psi(m)) = v(f \circ \psi)(m),$$
where $v \in T_mM$.

Proof. We must show two things.

1. $d\psi(v) \in T_{\psi(m)}N$, for $v \in T_mM$,
2. $d\psi$ is a linear mapping of vector spaces.

Proof of 1. We have to show that $d\psi(v)$ is a linear derivation on N at $\psi(m)$. That is

a. $d\psi(v)(f + \lambda g) = d\psi(v)(f) + \lambda d\psi(v)(g) \quad f, g \in C^\infty(N)$
b. $d\psi(v)(f \cdot g) = f d\psi(v)g + g d\psi(v)f$.

To see (a),
$$\begin{aligned} d\psi(v)(f + \lambda g)|_{\psi(m)} &= v((f + \lambda g) \circ \psi)(m) \\ &= v(f \circ \psi + \lambda g \circ \psi)(m) \\ &= v(f \circ \psi)(m) + \lambda v(g \circ \psi)(m) \quad \text{by (2.3), (2.4)} \\ &= d\psi(v)(f)(m) + \lambda d\psi(v)(g)(m). \end{aligned}$$

To see (b),
$$\begin{aligned} d\psi(v)(f \cdot g)|_{\psi(m)} &= v((f \cdot g) \circ \psi)(m) \\ &= v(f \circ \psi \cdot g \circ \psi)(m) \\ &= f \circ \psi \cdot v(g \circ \psi)(m) + v(f \circ \psi)(g \circ \psi)(m) \\ &\text{since } v \in T_mM \\ &= f(\psi(m)) \cdot d\psi(g)(m) + d\psi(f)(\psi(m) \cdot g(\psi(m)). \end{aligned}$$

Proof of 2. One has
$$\begin{aligned} d\psi(v_1 + \lambda v_2)(f) &= (v_1 + \lambda v_2)(f \circ \psi) \\ &= v_1(f \circ \psi) + \lambda v_2(f \circ \psi) \quad \text{by (2.3)} \\ &= d\psi(v_1)(f) + \lambda d\psi(v_2)(f). \quad \square \end{aligned}$$

The basic property of the map $d\psi$ is the following.

Proposition [2.4]. *(Chain Rule)*
Let M, N, X be C^∞ manifolds such that there is a commutative diagram of smooth maps
$$\begin{array}{ccc} M & \xrightarrow{\psi_1} & N \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ M & \xrightarrow[\psi_3]{} & X \end{array}$$
Then the diagram
$$\begin{array}{ccc} T_mM & \xrightarrow{d\psi_1} & T_{\psi_1(m)}N \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle d\psi_2} \\ T_mM & \xrightarrow[d\psi_3]{} & T_{\psi_3(m)}X \end{array}$$

commutes.

Proof. We have to show that for $v \in T_m M$

$$d\psi_3(v) = d\psi_2 \circ d\psi_1(v).$$

But,

$$d\psi_3(v) f(\psi_3(m)) = v(f \circ \psi_3)(m) = v(f \circ \psi_2 \circ \psi_1)(m).$$

Since

$$d\psi_1(v) \in T_{\psi_1(m)}N, \text{ then } d\psi_2(d\psi_1(v)) \in T_{\psi_2 \circ \psi_1(m)}X = T_{\psi_3(m)}X$$

with

$$\begin{aligned} d\psi_2(d\psi_1(v))(f)(\psi_3(m)) &= d\psi_1(v)(f \circ \psi_2)(\psi_1(m)) \\ &= v(f \circ \psi_2 \circ \psi_1)(m). \quad \square \end{aligned}$$

So now we know at least formally a way of moving between tangent spaces of manifolds. However, to understand $T_m M$ we need to relate it to what we know about, that is, to $T_x \mathbb{R}^n$. To do that we shall use a local coordinate chart (U, ϕ) around $m \in M$. So ϕ defines a diffeomorphism

$$\phi^{-1} : V \subset \mathbb{R}^n \longrightarrow U \subset M$$

and hence a differential vector space map

$$d\phi^{-1} : T_{\phi(m)}\mathbb{R}^n \longrightarrow T_m M.$$

We need next the following result.

Proposition [2.5]. *Let $\psi : M \to N$ be a diffeomorphism (or local diffeomorphism around m). Then*

$$d\psi : T_m M \longrightarrow T_{\psi(m)}N$$

is an isomorphism of vector spaces.

Proof. We know from [2.3] that $d\psi$ is a homomorphism, so we just need to show $d\psi$ is bijective. But that is easy, if ψ is a diffeomorphism then so is $\psi^{-1} : N \to M$ so $\psi \circ \psi^{-1} = \psi^{-1} \circ \psi = \text{id}$. Hence from [2.4]

$$d\psi \circ d\psi^{-1} = d\psi^{-1} \circ d\psi = d(\text{id}). \tag{2.5}$$

Because $d(\text{id})(v)(f) = v(f \circ \text{id}) = v(f) = I \cdot v(f)$ (I =identity vector space map) for all $f \in \mathbb{C}^\infty(M)$, $v \in T_m(M)$, then $d(\text{id}) = I$. So from (2.5) $d\psi$ has a 2-sided inverse, and that completes the proof. $\square$

In fact, there is a partial converse to [2.5] (which I shall not prove here).

Proposition [2.6]. *(Inverse Function Theorem)*
If $d\psi : T_m M \to T_{\psi(m)}N$ is an isomorphism, then ψ maps any sufficiently small open set U around $m \in M$ diffeomorphically to the open set $\psi(U)$ around $\psi(m)$ in N.

Corollary to Prop. [2..5]. *The linear map*

$$d\phi^{-1} : T_{\phi(m)}\mathbb{R}^n \to T_m M$$

is an isomorphism. In particular, $T_m M$ is an n-dimensional vector space. One has

$$T_m M = d\phi^{-1}(T_{\phi(m)}\mathbb{R}^n). \tag{2.6}$$

Proof. The only point in need of justification is (2.6). That is, we must show that $d\phi^{-1}(T_{\phi(m)}\mathbb{R}^n)$ does not depend on the 'parameterization' ϕ^{-1}. So let $\phi' : U' \to V'$ be a second chart around $m \in M$. Then $\phi' \circ \phi^{-1}$ maps some neighbourhood $\tilde{V}$ of $\phi(m)$ diffeomorphically onto a neighbourhood $\tilde{V}'$ of $\phi'(m)$. The commutative diagram of smooth maps between open sets

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\ \phi^{-1}\ } & U \cap U' \\ {\scriptstyle \text{id}}\downarrow & & \uparrow{\scriptstyle (\phi')^{-1}} \\ \tilde{V} & \xrightarrow[\phi' \circ \phi^{-1}]{\text{diffeomorphism}} & \tilde{V}' \end{array}$$

gives rise to a commutative diagram of linear maps

$$\begin{array}{ccc} TmM & \xrightarrow{\ \text{id}\ } & TmM \\ {\scriptstyle d\phi^{-1}}\uparrow\cong & & \cong\uparrow{\scriptstyle d\phi'^{-1}} \\ T_{\phi(m)}\mathbb{R}^n \cong \mathbb{R}^n & \xrightarrow[d(\phi' \circ \phi^{-1})]{\cong} & T_{\phi'(m)}\mathbb{R}^n \cong \mathbb{R}^n \end{array}$$

and

$$\text{image}(d\phi^{-1}) = \text{image}(d\phi'^{-1}).$$

Hence $T_m M$ is well-defined by (2.6) (it is independent of choice of ϕ). $\square$

However, what does change with different choices of chart $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ around m is the basis of $T_m M$. That is, for (U_1, ϕ_1), defining a coordinate system $(x_1, \ldots, x_n)$ around m, there is a canonical basis defined for $T_m M$ by

$$\boxed{\left.\frac{\partial}{\partial x_i}\right|_m = d\phi_1{}^{-1}\left(\left.\frac{\partial}{\partial u_i}\right|_{\phi_1(m)}\right)} \qquad i = 1, \ldots, n. \tag{2.7}$$

So for $f \in C^\infty(M)$,

$$\frac{\partial}{\partial x_i}\bigg|_m (f) = \frac{\partial}{\partial u_i}(f\circ\phi_1{}^{-1}\bigg|_{\phi_1(m)}.$$

(Note that because $\frac{\partial}{\partial u_i}\big|_{\phi_1(m)}$ is a basis for $T_{\phi_1(m)}\mathbb{R}^n$ then the $\frac{\partial}{\partial x_i}\big|_m$ are a basis for T_mM.)

We then have that for $\psi : M \to N$, if we choose coordinate charts (U_m, ϕ_m), (V_N, τ_N) around $m \in M$ and $\psi(m) \in N$, respectively, that T_mM and $T_{\psi(m)}N$ have canonical bases (or 'frames') and hence $d\psi : T_mM \to T_{\psi(m)}N$ is given by a specific matrix relative to those frames. To see the specific form of $d\psi$ the following lemma is very useful.

Lemma [2.7]. *Let $v \in T_mM$ then there exist uniquely numbers $v_1(m), \ldots, v_n(m)$ such that*

$$v = v_1(m)\,\frac{\partial}{\partial x_1}\bigg|_m + \cdots + v_n(m)\,\frac{\partial}{\partial x_n}\bigg|_m,$$

where $(x_1, \ldots, x_n)$ is the coordinate system defined by a chart (U, ϕ), and

$$v_i(m) = v(x_i). \tag{2.8}$$

Proof.

$$\begin{aligned} v(x_i) &= \sum_j v_j(m)\,\frac{\partial}{\partial x_j}\bigg|_m (x_i) = \sum_j v_j(m)\frac{\partial}{\partial u_j}(x_i\circ\phi^{-1}) \\ &= \sum_j v_j(m)\,\frac{\partial u_i}{\partial u_j}\bigg|_m = \sum_j v_j(m)\delta_{ij}(m) = v_i(m). \quad \square \end{aligned}$$

Proposition [2.8]. *Let $\psi : M \to N$ be as above with coordinate systems*

$$(U_m, x_1, \ldots, x_n) \qquad \text{and} \qquad (V_N, y_1, \ldots, y_d)$$

around $m \in M$ and $\psi(m) \in N$. Then relative to the canonical bases

$$d\psi\left(\frac{\partial}{\partial x_i}\bigg|_m\right) = \sum_{j=1}^{d} \frac{\partial}{\partial x_i}(y_j\circ\psi)\bigg|_m \frac{\partial}{\partial y_i}\bigg|_{\psi(m)}. \tag{2.9}$$

Proof. $d\psi\left(\frac{\partial}{\partial x_i}\big|_m\right) = \sum_j \alpha_j\,\frac{\partial}{\partial y_j}\big|_{\phi(m)}$ for some numbers α_j. Now just apply Lemma [2.7]. $\square$

The matrix $J = \left(\frac{\partial(y_i\circ\psi)}{\partial x_j}\right)$ is called Jacobian of the map ψ with respect to the given coordinate systems.

In summary, then we now have achieved a manifold analogue of the local statement (†) above. We have that the tangent space to $m \in M$ is identified locally as $T_mM = d\phi_i{}^{-1}(T_{\phi_i(m)}\mathbb{R}^n)$, where $\phi_i : U_i \to \mathbb{R}^n$ around m, independently of the choice of ϕ_i — what does change with different ϕ_i is the basis of T_mM. We obtain the basis $\frac{\partial}{\partial x_i}\big|_m = d\phi_i{}^{-1}\left(\frac{\partial}{\partial u_i}\big|_{\phi_i(m)}\right)$ with respect to (U_i, ϕ_i), so $\frac{\partial}{\partial x_i}\big|_m$ is a vector in T_mM "pulled-back" by $d\phi_i{}^{-1}$ from $T_{\phi_i(m)}\mathbb{R}^n$. To say it again, choosing different (U_i, ϕ_i) does not affect the identification of T_mM, but it does change the basis $\left\{\frac{\partial}{\partial x_1}\big|_m, \ldots, \frac{\partial}{\partial x_n}\big|_m\right\}$. Notice also that the derivative $\frac{\partial}{\partial x_i}\big|_m (f)$ of a function f on M is actually taken in $\mathbb{R}^n$. Because the ϕ_i are local diffeomorphisms and f depends on the patching functions $\phi_i\circ\phi_j{}^{-1}$ the idea is that this will reflect the topology of M.

For this to be a sensible definition we need to know that when we write $v \in T_mM$ using a basis of T_mM defined by a local chart (U_i, ϕ_i) then $v(f)|_m$ does not depend on that basis\chart. In other words, we need to know how the local representations are related —and they must be related precisely because $v(f)|_m$ is intrinsic. If we choose 2 coordinate charts (U_i, ϕ_i), (U_j, ϕ_j) around $m \in M$ with local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, so that we have two bases

$$\mathcal{E}_i = \left\{\frac{\partial}{\partial x_1}\bigg|_m, \ldots, \frac{\partial}{\partial x_n}\bigg|_m\right\} \quad \text{and} \quad \mathcal{E}_j = \left\{\frac{\partial}{\partial y_1}\bigg|_m, \ldots, \frac{\partial}{\partial y_n}\bigg|_m\right\}$$

for T_mM, then any $v \in T_mM$ can be written uniquely

$$v = \sum_{i=1}^n \xi_i\,\frac{\partial}{\partial x_i}\bigg|_m = \sum_{i=1}^n \eta_i\,\frac{\partial}{\partial y_i}\bigg|_m, \qquad \text{some } \xi_i, \eta_i \in \mathbb{R},$$

so

$$\sum_{k=1}^n \xi_k\,\frac{\partial}{\partial u_k}(f\circ\phi_i{}^{-1})\bigg|_m = \sum_{k=1}^n \eta_k\,\frac{\partial}{\partial u_k}(f\circ\phi_j{}^{-1})\bigg|_m$$

for $f \in C^\infty(M)$.

I leave it as a straightforward exercise to the reader (using Lemma [2.7]) to show that

$$\begin{pmatrix}\eta_1\\ \vdots \\ \eta_n\end{pmatrix} = \begin{pmatrix}\frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n}\\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n}\end{pmatrix}\begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix}.$$

That is, the change of basis matrix from $\mathcal{E}_i$ to $\mathcal{E}_j$ is the Jacobian

$$\left(\frac{\partial x_r}{\partial y_s}\right) = d(\phi_j \circ \phi_i^{-1})\big|_m .$$

Now let's talk about tangent bundles.

Well, so far these are all quite simple observations, involving not much more than elementary linear algebra. Nevertheless we have actually deduced some profound facts. This becomes clearer when we look at globally defined derivations, which are described by the tangent bundle of M,

$$TM = \bigcup_{m \in M} T_m M.$$

Whereas, $T_m M$ represents only local information about M, the tangent bundle TM contains global information, which is why we can deduce facts about the geometry and topology of M from TM. Explaining why TM is a manifold (of dimension $2n$) leads us to the prototypical construction of what is more generally called a 'vector bundle'. We proceed first by choosing a coordinate chart (U_1, ϕ_1) on M. Then we have the tangent bundle over U_1,

$$TU_1 = TM|_{U_1} = \bigcup_{m \in U_1} T_m M = \pi^{-1}(U_1), \tag{1}$$

where $\pi : TM \to M$ is the projection map that takes $v \in T_m M$ to $m \in M$ (ie it tells you which tangent space v is in). From our observations above we have a canonical diffeomorphism

$$\psi_1 : \pi^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R}^n. \tag{2}$$

For a point of TU_1 can be written (m, v), where $m = \pi(v)$, and the chart coordinates $(x_1, \ldots, x_n)$ identify uniquely n real numbers $\xi_1(m), \ldots, \xi_n(m)$ defining v, ie $v(m) = \sum_i \xi_i(m) \left.\frac{\partial}{\partial x_i}\right|_m$. So the map ψ_1 is defined by

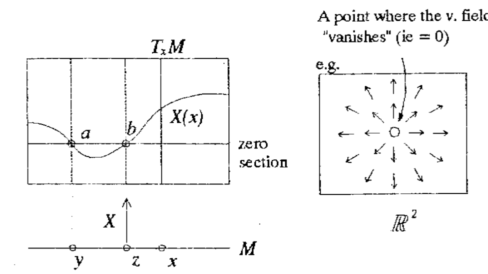
$$(m, v) \longmapsto (m, \xi_1(m), \ldots, \xi_n(m)).$$

(Check that it is a diffeomorphism!). Moreover, the identity (1) tells us precisely how the maps $\psi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{R}^n$ and $\psi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{R}^n$ are related over the intersection $U_1 \cap U_2$. That is, we have a map

$$U_1 \cap U_2 \longrightarrow GL(n; \mathbb{R})$$
$$m \longmapsto d(\phi_2 \circ \phi_1^{-1})\big|_m \stackrel{\text{def}}{=} g_{12}(m)$$

such that

$$\begin{pmatrix} \eta_1(m) \\ \vdots \\ \eta_n(m) \end{pmatrix} = g_{12}(m) \begin{pmatrix} \xi_1(m) \\ \vdots \\ \xi_n(m) \end{pmatrix}.$$



a and b are points where the vector field vanishes, ie $X(y)=(y,0)=a$; $X(z)=(z,0)=b$. This tell us about the topology of M.

The transformation g_{12} is called a 'transition function' by mathematicians and a 'local gauge transformation' by physicists. In particular, using the identification (2) we get a good C^∞ manifold structure on TM.

In summary then, the tangent bundle is a manifold TM canonically associated to M with a natural projection map $\pi : TM \to M$, and for each chart U_i on M there is a diffeomorphism $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$, and the different diffeomorphisms corresponding to different charts fit together over the intersection $U_i \cap U_j$ by the transformations $g_{ij} : U_i \cap U_j \to GL(n; \mathbb{R})$.

Notice that the transition functions g_{ij} satisfy

(3.1) $\quad g_{ik}(m) = g_{ij}(m) g_{jk}(m)$

(3.2) $\quad g_{ij}(m) = g_{ji}(m)^{-1}$

(3.3) $\quad g_{ii}(m) = \text{identity } I_n. \quad$ (follows from (3.1) and (3.2))

In fact, given C^∞ maps $S_i : U_i \to \pi_i(U_i)$ and g_{ij} satisfying the conditions (3.1) $\to$ (3.3) then we have the technology to rebuild TM. I'll leave it to the reader to think how to do that, but notice that the conditions (3.$*$) are essential compatibility conditions —why? (see below).

Now the point of talking about tangent bundles is to talk about vector fields. Recall that a vector field on $\mathbb{R}^n$ is an object like $X_{\mathbb{R}^n}(x) = \sum_{i=1}^n \alpha_i(x) \left.\frac{\partial}{\partial u_i}\right|_x$ —that is, it assigns to each $x \in \mathbb{R}^n$ a tangent vector $X_{\mathbb{R}^n}(x) \in T_x\mathbb{R}^n$, in a smooth way, which means that the functions $\alpha_i(x)$ are C^∞. A more sophisticated way to say that is that X is a smooth map $X_{\mathbb{R}^n} : \mathbb{R}^n \to T\mathbb{R}^n$ such that $\pi \circ X_{\mathbb{R}^n} = \text{Identity}$. We define a vector field on M in precisely the same way, ie a smooth map $X : M \to TM$ such that $\pi \circ X = \text{Identity}$. So X assigns 'smoothly' a vector $X(m) \in T_m M$ to each point m of M.

Locally this means that over (U_i, ϕ_i) X looks like

$$X(m) = \sum_{i=1}^n \alpha_i(m) \left.\frac{\partial}{\partial x_i}\right|_m ,$$

where $\alpha_i : U_i \to \mathbb{R}^1$ are C^∞ functions.

In "bundle" language X is called a (C^∞) section of TM; that is, a section of TM is a vector field on M.

Thus sections or vector fields have two equivalent descriptions:

(1) Global: A section $X : M \to TM$ of the tangent bundle.
(2) Local: A set of C^∞ functions $f_i : U_i \to \mathbb{R}^n$ (one for each chart) such that $f_j(x) = g_{ij}(x) f_i(x)$, for $x \in U_i \cap U_j$ and $g_{ij} = d(\phi_i \circ \phi_j{}^{-1})$ are the transition functions.

In understanding that point one has met with the basic idea of gauge theory. Indeed, the crucial point is the following. If we could cover M by a single coordinate chart then for certain any C^∞ vector field on M could be written as a C^∞ function $M \to \mathbb{R}^n$ defined with respect to the globally defined basis of TM induced by the coordinate chart. That is, the coordinate chart defines a basis $\left\{ \frac{\partial}{\partial x_i}\Big|_x \right\}$ in each $T_x M$ which varies in a smooth way with $x \in M$. Moreover, precisely because M is covered by a single chart then M is diffeomorphic to $\mathbb{R}^n$ —by definition. To put it another way, when it is not possible to write every C^∞ vector field on M as a C^∞ function then we know M has non-trivial topology. Thus M is topologically non-trivial when there is a "non-zero obstruction" to the existence of n globally defined and linearly independent vector fields $X_1, \ldots, X_n$ on M. That is, n smooth maps

$$X_i : M \longrightarrow TM$$

such that

$$\pi \circ X_i = \text{id} \qquad (\text{ie } X_i(x) \in T_x M)$$

and

$$\{ X_1(x), \ldots, X_n(x) \}$$

is a basis for $T_x M$, for each $x \in M$. If they exist, the $\{ X_1, \ldots, X_n \}$ are called a trivialization of TM —they can each be described locally with respect to the atlas $\{(U_i, \phi_i)\}$ of M. (Called a global gauge by physicists).

Locally we may always take $X_i(x) = \frac{\partial}{\partial x_i}\Big|_x$ with respect to the chart (U, ϕ), but the above conditions requires this to be possible globally. Notice in particular that each $X_i(x)$ must be non-zero, otherwise we will not get a basis for $T_x M$, and that is the way we construct the Euler number for M —ie the Euler number $\chi(M)$ counts the number of times a "generic" vector field hits the zero section (see the above picture). Thus if $\chi(M) \neq 0$ we know that it is impossible to find n linearly independent $\neq 0$ vector fields on M. $\chi(M)$ can be calculated in many different ways and serves as a crucial index for classifying manifolds of distinct topological type. This perhaps represents one of the most profound and beautiful areas of mathematics and its interaction with physics. Physicists refer to n global linearly independent sections as a global gauge for TM —thus a gauge transformation changes the global gauge to another global gauge, ie it changes (smoothly) the basis in each tangent space $T_x M$. The Euler number in particle physics parlance is called a topological quantum number.

In general, however, $\chi(M)$ is not sufficient on its own to tell if there exist n globally independent vector fields on M. That is $\chi(M) = 0 \;\not\Longrightarrow$ they exist —we only know this obstruction to their existence vanishes, so they may exist. There are $(n-1)$ other topological quantum numbers associated to M, called Chern classes, and we require all of those to be zero, to know their is no apriori obstruction. (Interesting fact: $\chi(M) = 0$ whenever $\dim(M)$ is odd —S^1 for example!) The mathematical term for "global gauge" is "(global) trivialization" of TM.

As a brief summary then, the idea is that observers in different coordinate patches can agree (in the overlap) that a tangent vector exists intrinsically and where it is, but usually they cannot agree how to describe it (ie what its coordinates are), though they do agree precisely how their different descriptions are related (ie the transition function).

So now let's look at some examples. The examples I'm going to give here are all essentially trivial, but I hope, nevertheless, instructive. First let's be clear about what finding a trivialization of the tangent bundle of a manifold means. It means

(1) Find n linearly independent C^∞ maps $M \xrightarrow{X_i} TM$ such that $\pi \circ X_i = \text{id}$ $(i = 1, \ldots, n,\ n = \dim M)$ or, equivalently, it means
(2) Giving linearly independent n-functions $f_i^{(a)} : U_a \to \mathbb{R}^n$ in each coordinate patch U_a such that $f_i^{(a)} = g_{ab} f_i^{(b)}$, where $g_{ab} : U_a \cap U_b \to GL(n;\mathbb{R})$ are the transition functions.

Notice that in (2) if in $U_a \cap U_b \cap U_c$ I give $f_i^{(a)}$, $f_i^{(b)}$ and $f_i^{(c)}$ such that

$$f_i^{(a)} = g_{ab} f_i^{(b)} \qquad \text{and} \qquad f_i^{(b)} = g_{bc} f_i^{(c)}$$

then I know that $f_i^{(a)}$ and $f_i^{(c)}$ also match-up —why? (Otherwise finding sections would in general be impossible).

A worthwhile task at this point is to work out exactly why (1) and (2) are the same thing. Of course, the conditions

$$f_i^{(a)} = g_{ab} f_i^{(b)} \qquad \text{on } U_a \cap U_b$$

are equations between column vectors in $\mathbb{R}^n$ and $g_{\alpha\beta} f_i^{(b)}$ is matrix multiplication. Note that each $f_i^{(a)}$ defines a map $U_a \to U_a \times \mathbb{R}^n$ with $\pi_a \circ f_i^{(a)} = \text{id}$ where $\pi_a : U_a \times \mathbb{R}^n \to U_a$ is the projection onto the first factor.

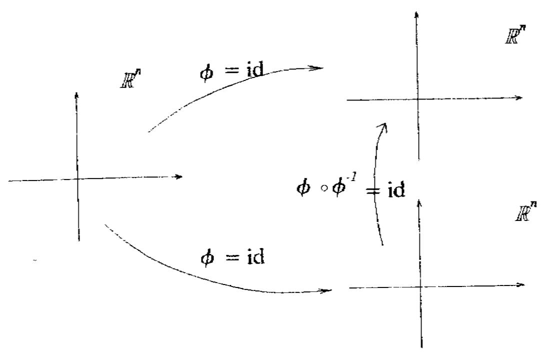
That means that each $f_i^{(c)}$ pulls back by the trivialization of $TU_a \cong U_a \times \mathbb{R}^n$ to a sector (v. field) of TU_a. These sections of TU_a agree on the overlap regions $U_a \cap U_b$ by the computability conditions on the g_{ab} and hence define a global section of TM (In particular, note that if $\tau_a : TU_a \to U_a \times \mathbb{R}^n$ is the trivialization then $\tau_a^{-1}(f_i^{(a)}) = \tau_b^{-1}(f_i^{(b)})$. Conversely, any section of TM has this form.

Now if we have a global trivialization we can write a vector field $X : M \to TM$ as a function $f_X : M \to \mathbb{R}^n$, defined independently of the coordinate charts —check this. However, notice that the function f_X depends explicitly on the choice of trivialization of TM. Changing the trivialization will change f_X. Thus the way we represent mathematical objects as functions depends on the underlying trivialization of the 'background' bundle —that is a deep fact. Usually we hope to find some canonical choice of trivialization that will associate a preferred function f_X that accords with the properties we might expect.

Anyway let's look at a couple of examples.

Example 1. $\mathbb{R}^n$ with a single chart $(\mathbb{R}^n, \mathrm{id})$.
We have essentially already dealt with this case above, but nevertheless it is instructive to do it directly.

For consistency let's think of this as follows:



The canonical basis for $T_m\mathbb{R}^n$ at $m \in \mathbb{R}^n$ is by definition

$$\frac{\partial}{\partial x_i}\bigg|_m = dI^{-1}\left(\frac{\partial}{\partial u_i}\bigg|_{I(m)}\right) = I\left(\frac{\partial}{\partial u_i}\bigg|_m\right) = \frac{\partial}{\partial u_i}\bigg|_m \qquad i = 1, \dots, n$$

where I have written $I = \mathrm{id}$. So, as we would expect, it is just the usual basis for $\mathbb{R}^n$ with coordinates (global) $x_i = u_i \circ I = u_i$. Given that our constructions are relative to $\mathbb{R}^n$ it would be discomforting if this were not the case. Anyway, we easily obtain the following result.

Theorem. *The tangent bundle $T\mathbb{R}^n$ of $\mathbb{R}^n$ is canonically trivial. That is, there exists a preferred global trivialization*

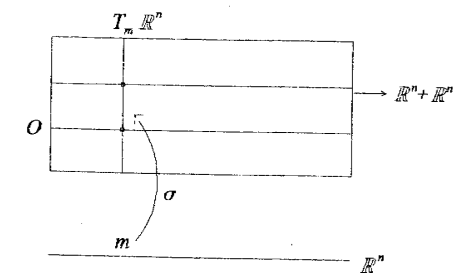
$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n.$$

Proof. Well, any element $v \in T\mathbb{R}^n$ can be written uniquely as

$$v = \sum_{i=1}^{n} \lambda_i(m) \frac{\partial}{\partial u_i}\bigg|_m,$$

where $\pi(v) = m$. So the desired map is given by

$$v \longmapsto (m, \lambda_1(m), \dots, \lambda_n(m)) \qquad (\text{note } m \in \mathbb{R}^n). \quad \square$$



In terms of sections we can define this trivialization by n-sections

$$\sigma_i(m) = \frac{\partial}{\partial u_i}\bigg|_m, \qquad m \in \mathbb{R}^n.$$

Thus under the above isomorphism, these elements of $T_m\mathbb{R}^n$ map to the canonical basis $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ of $\mathbb{R}^n$. This is the trivialization that is used in differential and integral calculus in $\mathbb{R}^n$. Of course, we could easily modify it to

$$\sigma_i^i(m) = f_i(m) \frac{\partial}{\partial u_i}\bigg|_m$$

where the f_i are C^∞ functions on $\mathbb{R}^n$. This means derivatives and integrals and so forth would have different (but correct) answers than the usual ones, but the two answers corresponding to the two different trivializations are precisely related by the transition functions —try some examples!

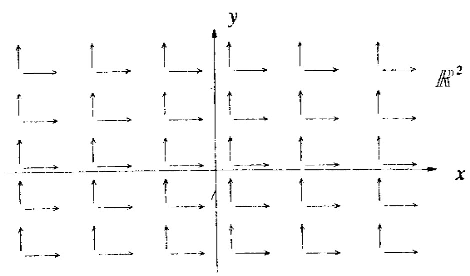
The above trivialization of $T\mathbb{R}^n$ is the natural choice because $\{(\mathbb{R}^n, \mathrm{id})\}$ is the natural atlas for $\mathbb{R}^n$. But because we want to see our general constructions in action we could consider $\mathbb{R}^n$ with the most general atlas —given that $\mathbb{R}^n$ is endowed with the metric topology. So we take the atlas

Example 2. $\mathcal{A} = \{ (U_a, \phi_a) : a \in \mathbb{R}^+ \}$
where $\mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \}$ and $U_a = \{ x \in \mathbb{R}^n : ||x|| < a \}$ with $|| \cdot ||$ the Euclidean norm, and $\phi_a : U_a \longrightarrow \mathbb{R}^n$ is a homeomorphism such that $\phi_a \circ {\phi_b}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism $\forall\, a, b \in \mathbb{R}^+$. Initially it may seem a rather formidable problem to look for a trivialization in this case, but actually it's very easy. Think about $\mathbb{R}^n$ geometrically: it's not hard to see that the tangent bundle can be trivialized by choosing n-vector fields parallel to the axes. E.g. for $\mathbb{R}^2$

To make that precise we can define such vector fields by identifying the flow lines. Take the map, for example,

$$\sigma_\epsilon^{(i)} : \mathbb{R} \longrightarrow \mathbb{R}^n \quad \epsilon = (\epsilon_1, \dots, \hat{\epsilon}_i, \dots, \epsilon_n) \in \mathbb{R}^{n-1} \quad (\hat{}\ \text{means "omited"})$$

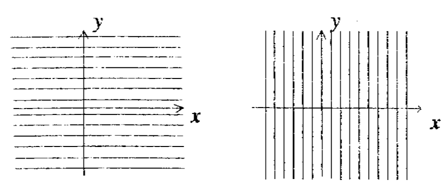
$$\sigma_\epsilon^{(i)}(t) = (\epsilon_1, \epsilon_2, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}\ \text{position.}}}{t}, \dots, \epsilon_n)$$



This is the line in $\mathbb{R}^n$ parallel to the i^{th} coordinate axis passing through the point

$$(\epsilon_1, \epsilon_2, \ldots, \underset{i^{\text{th}}\ \text{pos}^{\underline{n}}.}{\underset{\uparrow}{0}}, \ldots, \epsilon_n).$$

For $n = 2$, these are the lines:



for $i=1$, and for $i=2$.

Now define the corresponding vector field along $\sigma_\epsilon^{(i)}$ by

$$v_{\sigma_\epsilon^{(i)}(t)} \in T_{\sigma_\epsilon^{(i)}(t)}\mathbb{R}^n, \quad V_{\sigma_\epsilon^{(i)}(t)} = d\sigma_\epsilon^{(i)} \underbrace{\left(\frac{d}{dt}\bigg|_t\right)}_{\text{derivate along } \mathbb{R}^1.}$$

So we obtain a vector field on $\mathbb{R}^n$,

$$v^{(i)} : \mathbb{R}^n \longrightarrow T\mathbb{R}^n, \qquad v^{(i)}(m) = v_{\sigma_\epsilon^{(i)}(t)} \in T_m\mathbb{R}^n$$

where t, ϵ are the unique numbers such that $m = \sigma_\epsilon^{(i)}(t)$.

It is also easy to show that this defines n-linearly independent v. fields ie

$$\{\, v^{(1)}(m), \ldots, v^{(n)}(m) \,\}$$

is a basis for $T_m\mathbb{R}^n$.

Exercise. Show it. (In fact, $v^{(k)}(m)(f) = \frac{\partial f}{\partial u_k}\Big|_m$)
Because the curves are C^∞ this defines n linearly independent C^∞ maps

$$v^{(i)} : M \longrightarrow TM, \quad M = \mathbb{R}^n$$

and hence $T\mathbb{R}^n$ is trivial. Notice that the only point at which we have used the manifold structure of $\mathbb{R}^n$ is in the assertion that the curves are C^∞.

Exercise. Show the curves $\sigma_\epsilon^{(i)} : \mathbb{R}^1 \to \mathbb{R}^n$ <u>are</u> C^∞ with respect to the manifold structure $\mathcal{A}$.
Now we can also see how this works with respect to the local trivializations

$$\begin{aligned} TU_a &\longrightarrow U_a \times \mathbb{R}^n \\ v &\longmapsto (m, \alpha_1(m), \ldots, \alpha_n(m)) \end{aligned}$$

where $\pi(v) = m$, and

$$v = \sum_{m=1}^{n} \alpha_i(m)\, \frac{\partial}{\partial x_i}\bigg|_m \quad \text{with} \quad \frac{\partial}{\partial x_i}\bigg|_m = d\phi_a{}^{-1}\left(\frac{\partial}{\partial u_i}\bigg|_{\phi_a(m)}\right).$$

In the overlap $U_a \cap U_b$ besides the basis $\left\{\frac{\partial}{\partial x_i}\Big|_m\right\}$ for $T_m\mathbb{R}^n$ there is also the basis

$$\left\{\frac{\partial}{\partial y_i}\bigg|_m = d\phi_b{}^{-1}\left(\frac{\partial}{\partial u_i}\bigg|_{\phi_b(m)}\right)\right\}.$$

So we can write the vectors $v^{(i)}(m) \in T_m\mathbb{R}^n$ in two different ways:

$$\begin{aligned} &v^{(i)}(m) = \sum \xi_j(m)\, \tfrac{\partial}{\partial x_j}\Big|_m \\ \text{and} \qquad & \qquad\qquad\qquad\qquad \text{for some } \xi_j(m), \zeta_j(m) \in \mathbb{R}^1. \\ &v^{(i)}(m) = \sum \zeta_j(m)\, \tfrac{\partial}{\partial y_j}\Big|_m \end{aligned}$$

So we have maps

$$\begin{aligned} &U_a \xrightarrow{f_a} \mathbb{R}^n \qquad m \longmapsto (\xi_1(m), \ldots, \xi_n(m)) \\ \text{and} \\ &U_b \xrightarrow{f_b} \mathbb{R}^n \qquad m \longmapsto (\zeta_1(m), \ldots, \zeta_n(m)) \end{aligned}$$

and, by <u>definition</u> of $g_{ab} = d\phi_a \circ d\phi_b{}^{-1}$, we have in $U_a \cap U_b$

$$f_a(m) = g_{ab}(m) f_b(m),$$

as required. So from either the direct point of view, or by doing it locally we have

Theorem (2). *The tangent bundle $T\mathbb{R}^n$ with the atlas $\mathcal{A}$ for $\mathbb{R}^n$ is canonically trivial, ie there is a preferred diffeomorphism*

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n.$$

Of course, we can change the trivialization by multiplying the sections $v^{(i)}$ by smooth non-zero functions.

For interest we could take the case, for example, of $\mathbb{R}^1$ with 2 charts $U_1 = (-\infty, 1)$, $U_2 = (-1, \infty)$, so $U_1 \cap U_2 = (-1, 1)$.

Over U_1 we have a trivialization (for $v(m) \neq 0$)

$$TU_1 \longrightarrow U_1 \times \mathbb{R}^1, \ v(m) \longmapsto (m, \alpha(m)), \text{ where } v(m) = \alpha(m) \left.\frac{\partial}{\partial x}\right|_m,$$

over U_2 we have a trivialization

$$TU_2 \longrightarrow U_2 \times \mathbb{R}^1, \ v(m) \longmapsto (m, \beta(m)), \text{ where } v(m) = \beta(m) \left.\frac{\partial}{\partial y}\right|_m,$$

and here $\left.\frac{\partial}{\partial x}\right|_m = d\phi_1{}^{-1}\left(\left.\frac{\partial}{\partial u}\right|_{\phi_1(m)}\right)$, $\left.\frac{\partial}{\partial y}\right|_m = d\phi_2{}^{-1}\left(\left.\frac{\partial}{\partial u}\right|_{\phi_2(m)}\right)$. Clearly then the transition function must be

$$g_{12} = \frac{\alpha(m)}{\beta(m)} \qquad (\text{note } \beta \neq 0).$$

To see that fits with our theory note that

$$v(m) = \alpha(m) \left.\frac{\partial}{\partial x}\right|_m = \beta(m) \left.\frac{\partial}{\partial y}\right|_m = \beta(m) \left.\frac{\partial x}{\partial y}\right|_m \left.\frac{\partial}{\partial x}\right|_m.$$

So,

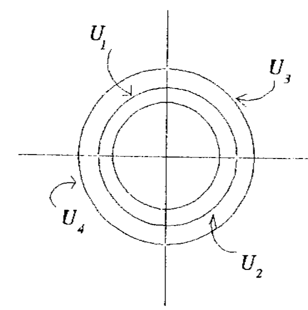
$$\alpha(m) = \beta(m) \left.\frac{\partial x}{\partial y}\right|_m, \text{ ie } \frac{\alpha(m)}{\beta(m)} = \left.\frac{\partial x}{\partial y}\right|_m = g_{12}(m) \quad (\text{ie } d\phi_1 \circ d\phi_2{}^{-1})$$

as usual.

Try this with some test functions, e.g. $\phi_1(m) = m^2$, $\phi_2(m) = e^{-m}$. (The reader should find the above just amounts to the change of variable formula (ie Chain Rule) in 1-dimension —what are $\left.\frac{\partial}{\partial x}\right|_m$ and $\left.\frac{\partial}{\partial y}\right|_m$??)

Now let's look at the simplest topologically non-trivial manifold.

Example 3. The unit circle S^1.
Of course $S^1 = \{\underline{x} \in \mathbb{R}^2 : \|\underline{x}\| = 1\}$. There are many ways to give S^1 an atlas, but here I shall move between the following 2:



Atlas 1

$$\begin{aligned} U_1 &= \{(x, \sqrt{1-x^2}) : x \in (-1,1)\}, & \phi_1(x,y) &= x \\ U_2 &= \{(x, -\sqrt{1-x^2}) : x \in (-1,1)\}, & \phi_2(x,y) &= x \\ U_3 &= \{(\sqrt{1-y^2}, y) : y \in (-1,1)\}, & \phi_3(x,y) &= y \\ U_4 &= \{(-\sqrt{1-y^2}, y) : y \in (-1,1)\}, & \phi_4(x,y) &= y. \end{aligned}$$

Atlas 2

$$\begin{aligned} V_1 &= \{(\cos t, \sin t) : t \in (0, 2\pi)\}, & \psi_1(\cos t, \sin t) &= t \\ V_2 &= \{(\cos t, \sin t) : t \in (-\tfrac{\pi}{2}, \tfrac{3\pi}{2})\}, & \psi_2(\cos t, \sin t) &= t. \end{aligned}$$

Well perhaps the first task is to explain why these atlases define a manifold structure on S^1. But that is clear, because, for example, in $U_1 \cap U_3$ one has $\phi_1 \circ \phi_3{}^{-1}(y) = \sqrt{1-y^2}$ which is certainly C^∞. Also $(\phi_1 \circ \phi_3{}^{-1})^{-1}(x) = \phi_3 \circ \phi_1{}^{-1}(x) = \sqrt{1-x^2}$ is C^∞ and hence $\phi_1 \circ \phi_3{}^{-1}$ is a diffeomorphism (since obviously it is a homeomorphism). Similarly, the other transition functions are C^∞ diffeomorphisms and hence S^1 is a C^∞-manifold. (For atlas 2 it is even easier to see S^1 has a manifold structure.)

Notice also that S^1 also has a natural group structure under rotations and reflections. That is, if we write $e^{it} = (\cos t, \sin t)$ then it is clear that

$$e^{it} \cdot e^{is} = e^{i(t+s)} = (\cos(t+s), \sin(t+s)) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \in S^1,$$

and also that $\overline{e^{it}} = e^{-it} \in S^1$.

With both group and manifold structure S^1 is written $U(1)$, which is the simplest example of a compact connected Lie group (since the group action is C^∞). By definition, $U(1) = \{z \in \mathbb{C} \setminus \{0\} : |z| = 1\}$. In fact, the group action is not only C^∞ but it also has C^∞ inverse. Another way to say that, is that

$$e^{it} : S^1 \longrightarrow S^1 \qquad (U(1) \longrightarrow U(1))$$

is a diffeomorphism. To see that is true we must show that

$$\phi_i f \phi_j^{-1} \quad \text{and} \quad \phi_i f^{-1} \phi_j^{-1}$$

are C^∞ functions $\mathbb{R}^1 \longrightarrow \mathbb{R}^1$ for all $i, j = 1, 2, 3, 4$. I leave it to the reader to check that elementary fact. Let $\text{Diff}(S^1)$ denote the space of all diffeomorphisms of S^1. This is an infinite-dimensional Lie group and by our observations we have:

Proposition. *$U(1)$ is a subgroup of* Diff S^1.

In fact, the quotient space Diff $S^1/U(1)$ plays an important role in conformal field theory and in string theory in theoretical physics.

(To see Diff S^1 is infinite-dimensional note that any element for this group has a Fourier series descomposition $f = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} = \sum_{n=0}^{\infty} a_n \cos n\theta + b_n \sin n\theta$ for suitable a_k, b_k, c_k, so requires an infinite basis $\{e^{ik\theta}\}$.)

Now since S^1 is a C^∞ manifold then it has a tangent bundle TS^1 associated to it. We have

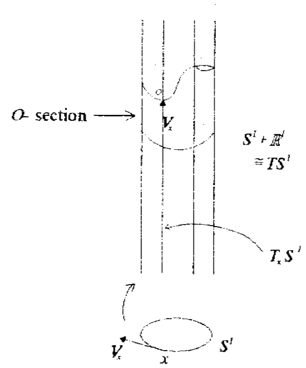
Theorem 3. *TS^1 is trivial. That is, there is a diffeomorphism*

$$TS^1 \cong S^1 \times \mathbb{R}^1.$$

Proof. Perhaps the first question to answer is, what is $T_x S^1$? Well, we saw in the theory that

$$T_x S^1 = d\phi_i^{-1}(T_{\phi_i(x)}\mathbb{R}^1)$$

where $x \in U_i$, and that this is independent of ϕ_i. So we could check that directly in this case.



Graph of a trivialization. $a \in T_x S^1$ defines a 'generator' for the line $T_x S^1$, ie a coordinate —any element of $T_x S^1$ can be given a coordinate relative to it, (a is the lenght of tangent vector V_x at x).

Let's do this for the second atlas to begin with. Take $\theta \in (0, \frac{\pi}{2})$. Then

$$\psi_1^{-1}(\theta) = (\cos\theta, \sin\theta)$$

and

$$d\psi_1^{-1} : T_\theta\mathbb{R}^1 \longrightarrow T_{(\cos\theta, \sin\theta)}S^1.$$

Well, a typical element of $T_\theta\mathbb{R}^1$ has the form

$$V_\theta = \alpha \left.\frac{d}{d\theta}\right|_\theta$$

and $d\psi_1^{-1}$ takes this to the element

$$d\psi_1^{-1}\left(\alpha\frac{d}{d\theta}\right) \in T_{(\cos\theta, \sin\theta)}S^1.$$

This acts on $f \in C^\infty(S^1)$ by

$$d\psi_1^{-1}\left(\alpha\frac{d}{d\theta}\right)(f) = \alpha\frac{d}{d\theta}f(\cos\theta, \sin\theta). \qquad (*)$$

To see what's going on geomectrically note that because we have defined S^1 as a subset of $\mathbb{R}^2$ then we can express any element of $T_{(\cos\theta, \sin\theta)}S^1$ in terms of the cannonical basis vector $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$ of $T_{(\cos\theta, \sin\theta)}\mathbb{R}^2$ ($u_1 = x$, $u_2 = y$). That is,

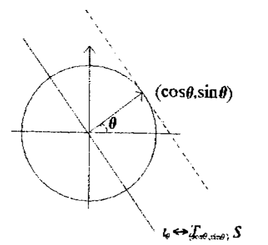
$$d\psi_1^{-1}\left(\alpha\frac{d}{d\theta}\right) = \lambda_1\frac{\partial}{\partial u_1} + \lambda_2\frac{\partial}{\partial u_2}.$$

To find the numbers λ_1, λ_2 we first, as usual (Lemma [2.7]) plug in u_1 and u_2; so for λ_1,

$$\begin{aligned}
\lambda_1 = d\psi_1^{-1}\left(\alpha\frac{d}{d\theta}\right)(u_1) &= \alpha\frac{d}{d\theta}(u_1 \circ \psi_1^{-1}) \\
&= \alpha\frac{d}{d\theta}(u_1(\cos\theta, \sin\theta)) \\
&= \alpha\frac{d}{d\theta}(\cos\theta) \text{ (by def}^{\underline{n}} \text{ of } u_1 \text{ as a function)} \\
&= -\alpha\sin\theta.
\end{aligned}$$

Similarly, $\lambda_2 = \alpha\cos\theta$. So

$$d\psi_1^{-1}\left(\alpha\frac{d}{d\theta}\right) = \alpha\left(-\sin\theta\frac{\partial}{\partial u_1} + \cos\theta\frac{\partial}{\partial u_2}\right).$$



The relation with (∗) is simply the Chain rule (implicit differentiation). Now because θ is a constant any vector in $T_{(\cos\theta,\sin\theta)}S^1$ is determined by the number α (its length). What θ tell's us is the direction of the tangent space (line) in $\mathbb{R}^2$. That is, the tangent line is the line

$$\iota_\theta = \{\, \alpha(-\sin\theta, \cos\theta) : \alpha \in \mathbb{R}^1 \,\}$$

which is the line in $\mathbb{R}^1$ parallel to S^1 at θ and passing through 0.

So that makes good sense. Note that ι_θ is independent of ψ_1: we get precisely the same line with ψ_2.

The reader may like to do the same thing for the first atlas in (x, y) coordinates. You will get

$$d\phi_1{}^{-1}\left(\alpha\frac{\partial}{\partial u}\right) = \alpha\left(\frac{\partial}{\partial u_1} - \frac{x}{\sqrt{1-x^2}}\frac{\partial}{\partial u_2}\right) = \alpha\frac{\partial}{\partial x}$$

$$d\phi_2{}^{-1}\left(\alpha\frac{\partial}{\partial u}\right) = \alpha\left(-\frac{y}{\sqrt{1-y^2}}\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right) = \alpha\frac{\partial}{\partial y}.$$

(The tangent line is $\iota_{(x,y)} = \{\, \alpha\left(1, \frac{-x}{\sqrt{1-x^2}}\right) : \alpha \in \mathbb{R}^1 \,\}$ which you can see is the same line as ι_θ for corresponding θ.)

Because the transition function is

$$g_{12}(y) = d(\phi_1 \circ \phi_2{}^{-1}) = -\frac{y}{\sqrt{1-y^2}} = -\frac{y}{x}$$

we see that

$$g_{12}(y)d\phi_1{}^{-1}\left(\alpha\frac{\partial}{\partial u}\right) = d\phi_2{}^{-1}\left(\alpha\frac{\partial}{\partial u}\right)$$

ie

$$g_{12}(y)\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$$

—as predicted by the theory.

So it remains to trivialize TS^1. The easiest way to do this is to observe that the vector $\binom{-y}{x}$ is orthogonal (in the Euclidean sense) to the vector $\binom{x}{y}$. So simply define the vector field

$$v(x,y) = -y\frac{\partial}{\partial u_1} + x\frac{\partial}{\partial u_2}.$$

Locally: For U_1, this is

$$v(x,y) = -\sqrt{1-x^2}\frac{\partial}{\partial u_1} + x\frac{\partial}{\partial u_2} = -\sqrt{1-x^2}\frac{\partial}{\partial x}.$$

For U_3, this is

$$v(x,y) = -y\frac{\partial}{\partial u_1} + \sqrt{1-y^2}\frac{\partial}{\partial u_2} = \sqrt{1-y^2}\frac{\partial}{\partial y}.$$

Since the local transition function between U_1 and U_3 is

$$g_{13} = -\frac{y}{x} = -\frac{\sqrt{1-x^2}}{x} = -\frac{y}{\sqrt{1-y^2}}$$

we see that the local functions

$$f_1 : U_1 \longrightarrow \mathbb{R}^1, \qquad f_1(x) = -\sqrt{1-x^2}$$
$$f_3 : U_3 \longrightarrow \mathbb{R}^1, \qquad f_3(y) = \sqrt{1-y^2}$$

give the local version of this trivialization.

Because they are C^∞ functions and v is C^∞ (why is v C^∞?) then this will do nicely. □

Observe that now <u>any</u> vector field $\tilde{v}$ (C^∞) on S^1 can be written

$$\tilde{v} = f_{\tilde{v}} \cdot v \qquad \text{ie } \tilde{v}(x,y) = f_{\tilde{v}}(x,y) \cdot v(x,y)$$

for some C^∞ function $f_{\tilde{v}}$ on S^1. So <u>relative</u> to the trivialization v the vector field is uniquely represented by the function f_v. That is what we mean by a global trivialization and that is what we mean by a global gauge for TS^1.

Exercise. Let $[\iota] \in \mathbb{R}P^1$, so $[\iota]$ is the point representing a line ι in $\mathbb{R}^2$. Show that

(i) $T_{[\iota]}\mathbb{R}P^1 \cong \mathrm{Hom}(\iota, \iota^{\perp})$.
(ii) $T\mathbb{R}P^1 \cong \mathbb{R}P^1 \times \mathbb{R}^1$ (ie trivial). Try doing this directly first, then deduce it from the diffeomorphism $\mathbb{R}P^1 \to S^1$ we constructed earlier.
(iii) Deduce that $T(S^1 \times S^1)$ is trivial by constructing an explicit trivialization using the trivialization of TS^1 given.

In summary, the tangent bundle to M is a manifold TM with canonical projection map $\pi : TM \to M$, such that for each local chart U_i on M there is a diffeomorphism

$$\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n \qquad \pi^{-1}(U_i) = TM|_{U_i}$$

and on restriction to $m \in U_i$,

$$\psi_i|_m : \pi^{-1}(m) = T_m M \longrightarrow \mathbb{R}^m \times \{m\} \xrightarrow{\mathrm{proj}^{\mathbb{R}}} \mathbb{R}^n$$

is a vector space isomorphism. The different diffeomorphisms correspond to different charts fit together over the intersections $U_i \cap U_j$ by the gauge transformations

$$g_{ij} : U_i \cap U_j \longrightarrow GL(n;\mathbb{R}). \qquad \text{(Gauge group)}$$

We note that these transformations satisfy the following consistency relations

(1) $g_{ij}(m)g_{jk}(m)g_{ki}(m) = \mathrm{I}$ identity on $U_i \cap U_j \cap U_k$
(2) $g_{ij}(m) = g_{ji}(m)^{-1}$ (so $g_{ii(m)=\mathrm{I}}$)

In fact, given g_{ij} on each $U_i \cap U_j$ is enough to define TM. (*)
Sections of TM, or vector fields on M have two equivalent descriptions:
1. Global: a vector field is a section $X : M \to TM$ (X smooth and $\pi \circ X = \mathrm{id}$)
2. (*) Local: a vector field is given by a set of C^∞ vector valued functions $f_i : U_i \to \mathbb{R}^n$ such that $f_j(m) = g_{ij}(m) f_i(m)$, for $m \in U_i \cap U_j$ where $g_{ij}(m) = d(\phi_j \circ \phi_i^{-1})$.

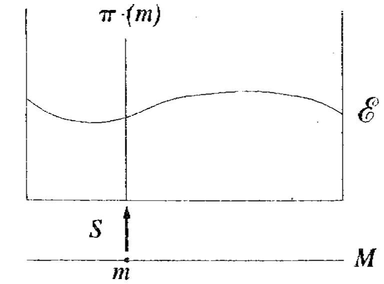
2. Vector Bundles

The tangent bundle is a very special bundle in that it is completely determined by M, it is intrinsic to M. That is why it is the basic object of interest in general relativity which is concerned with the geometry of M, and the geometry of M is encoded in TM. However, in quantum field theory one is interested in mathematically similar objects to TM, but which are ex-trinsic to M, they carry aditional data, corresponding to the 'internal symmetries' of a particle moving in M.

Definition. Let $\pi : \mathcal{E} \to M$ be a surjective map of manifolds whose fibre $\pi^{-1}(m)$ is a vector space for each $m \in M$.

The map π is called a C^∞ real vector bundle of rank n if there is an open cover $\{U_\alpha\}$ of M and fibre-preserving diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) = \mathcal{E}_{|U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n \tag{3.1}$$



which are linear isomorphisms on each fibre, ie

$$\begin{array}{ccccc} \pi^{-1}(m) & \xrightarrow{\psi_\alpha(m)} & \{m\} \times \mathbb{R}^n & \xrightarrow{\mathrm{proj}^{\mathbb{R}}} & \mathbb{R}^n \\ \downarrow \mathrm{id} & & & & \uparrow g_{\alpha\beta} \\ \pi^{-1}(m) & \xrightarrow{\psi_\beta(m)} & \{m\} \times \mathbb{R}^n & \xrightarrow{\mathrm{proj}^{\mathbb{R}}} & \mathbb{R}^n \end{array}$$

is a vector space isomorphism.
One often calls this construction a vector bundle $\mathcal{E}$ over M.

It is immediate from the summary in sect. 2 that TM is a vector bundle with fibre $\pi^{-1}(m) = T_m M$. As with TM a vector bundle $\mathcal{E} \xrightarrow{\pi} M$ has transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow GL(n;\mathbb{R}) \tag{3.3}$$

defined by

$$g_{ij}(m) = \psi_i(m) \circ \psi_j(m)^{-1} : \mathbb{R}^n \to \mathbb{R}^n. \tag{3.4}$$

They satisfy

$$\begin{cases} g_{ij} \cdot g_{jk} \cdot g_{ki} = \mathrm{I} & \text{on } U_i \cap U_j \cap U_k \\ g_{ii} = \mathrm{I} & \text{on } U_i. \end{cases} \tag{3.5}$$

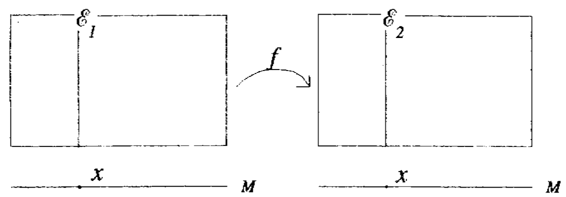
In fact, given maps (3.3) satisfying (3.5) one may construct a vector bundle with these transition functions. [Roughly: one defines

$$\mathcal{E} = E/G$$

where $E = M \times \mathbb{R}^n = \bigcap_i U_i \times \mathbb{R}^n$ and G acts on E by defining an equivalence relation by

$$(x, v) \sim (y, w) \qquad (x, v) \in U_i \times \mathbb{R}^n,\ (y, w) \in U_j \times \mathbb{R}^n$$

if and only if



$$y = x \quad \text{and} \quad w = g_{ij}(x)v.]$$

If we relax our assumption that the fibre $\pi^{-1}(m)$ is a real vector space and suppose that $\pi^{-1}(m)$ is isomorphic to $\mathbb{C}^n$, then $\mathcal{E}$ is said to be a complex vector bundle of rank n.

A smooth section of $\mathcal{E}$ is given by a smooth map

$$s : M \longrightarrow \mathcal{E}$$

such that

$$\pi \circ s = \text{id} \qquad (\text{so } s(m) \in \mathcal{E}_m = \pi^{-1}(m)).$$

We denote the space of all such smooth sections of $\mathcal{E}$ by $C^\infty(M;\mathcal{E})$. An element of $C^\infty(M;\mathcal{E})$ can be described locally by functions

$$f_i : U_i \longrightarrow \mathbb{R}^n$$

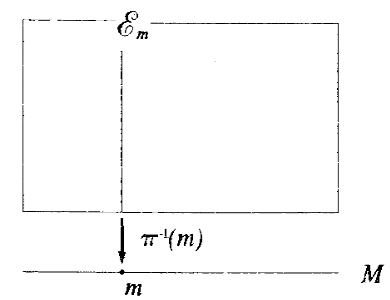
such that

$$f_j(m) = g_{ij}(m) f_i(m).$$

Exercise. Prove it.
In particular, if we can arrange that all $g_{ij}(m) =$ identity then each section can be written globally as a function. [$g_{ij} = I \implies f_i = f_j$ over $U_i \cap U_j \in GL(n;\mathbb{R})$]

In explaining what is meant by "arrange" we meet the basic mathematical idea of "gauge transformation". The point is that we do not really regard isomorphic vector bundles as the same.

A homomorphism between two bundles $\mathcal{E}_1, \mathcal{E}_2 \to M$, called a bundle map, is a C^∞ map $f : \mathcal{E}_1 \to \mathcal{E}_2$ which is a fibre-preserving map and restricts to a homomorphism



$f_{|_x} : (\mathcal{E}_1)_x \to (\mathcal{E}_2)_x$. f is a bundle diffeomorphism if f is a diffeomorphism and $f_{|_x}$ is an isomorphism of vector spaces; then f defines an isomorphism of vector bundles.

If $\mathcal{E}_1 = \mathcal{E}_2$ then we may formally regard an isomorphism of $\mathcal{E}$ with itself as an element of

$$\mathcal{G}(\mathcal{E}) = C^\infty(M, \text{Aut}(\mathcal{E}))$$

(If $\mathcal{E} \cong M \times \mathbb{R}^n$, then this is $C^\infty(M, GL(n;\mathbb{R}))$.) where $\pi : \text{Aut}(\mathcal{E}) \to M$ is the bundle associated to $\mathcal{E}$ with fibre

$$\boxed{\text{Aut}(\mathcal{E}_m) = \{\text{space of automorphisms of vector space } \mathcal{E}_m\}}.$$

This is actually a bundle of (Lie) groups, not a vector bundle, though I shall not justify the bundle structure here, $\mathcal{G}(\mathcal{E})$ is called the space of gauge transformations of $\mathcal{E}$. For practical purposes, an element $g \in \mathcal{G}(\mathcal{E})$ is something that assigns a C^∞ map

$$g_\alpha : U_\alpha \longrightarrow GL(n;\mathbb{R})$$

to each coordinate patch U_α of M over which $\mathcal{E}$ is trivialized, (and this is all we will need to know) such that the bundle defined by the transition functions

$$\widetilde{g_{\alpha\beta}} = g_\alpha g_{\alpha\beta} {g_\beta}^{-1} : U_\alpha \cap U_\beta \longrightarrow GL(n;\mathbb{R})$$

is isomorphic to $\mathcal{E}$. In fact, we have the following

Proposition. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be vector bundles over M. Then $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if there are smooth maps*

$$g_\alpha : U_\alpha \longrightarrow GL(n;\mathbb{R})$$

such that

$$g^{(1)}_{\alpha\beta} = g_\alpha g^{(2)}_{\alpha\beta} {g_\beta}^{-1}$$

where $g^{(k)}_{\alpha\beta}$ denote the transition functions of $\mathcal{E}_k$.

Proof. This is an immediate consequence of the following diagram ($U_{\alpha\beta} = U_\alpha \cap U_\beta$):

$$\begin{array}{ccccccc}
\pi^{-1}(U_{\alpha\beta}) & = & \pi^{-1}(U_{\alpha\beta}) & \underset{(?)}{\overset{\cong}{\longrightarrow}} & \pi_2^{-1}(U_{\alpha\beta}) & = & \pi_2^{-1}(U_{\alpha\beta}) \\
\psi_\alpha \downarrow & & \downarrow \psi_\beta & & \phi_\beta \downarrow & & \downarrow \phi_\alpha \\
U_{\alpha\beta} \times \mathbb{R}^n & & U_{\alpha\beta} \times \mathbb{R}^n & \underset{g_\beta}{\longleftarrow} & U_{\alpha\beta} \times \mathbb{R}^n & & U_{\alpha\beta} \times \mathbb{R}^n \\
\| & & \downarrow g_{\alpha\beta} & & \widetilde{g_{\alpha\beta}} \downarrow & & \| \\
U_{\alpha\beta} \times \mathbb{R}^n & = & U_{\alpha\beta} \times \mathbb{R}^n & \underset{g_\alpha}{\longleftarrow} & U_{\alpha\beta} \times \mathbb{R}^n & = & U_{\alpha\beta} \times \mathbb{R}^n
\end{array}$$

That is, there is a bundle isomorphism if and only if the rectangle conmutes, ie

$$g_{\alpha\beta} g_\beta = g_\alpha \widetilde{g_{\alpha\beta}}$$

so that

$$g_{\alpha\beta} = g_\alpha \widetilde{g_{\alpha\beta}} {g_\beta}^{-1}. \quad \square$$

Notice in particular that if we can find $g_\alpha : U_\alpha \longrightarrow GL(n;\mathbb{R})$ such that

$$\widetilde{g_{\alpha\beta}} = g_\alpha g_{\alpha\beta} {g_\beta}^{-1} = \text{identity}$$

then $\mathcal{E}$ is trivial (and vice-versa) —because then all the local trivializations or gauges, $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n$ match up over the intersections $U_\alpha \cap U_\beta$ and hence define a global gauge

$$\pi^{-1}(\bigcup_\alpha U_\alpha) \cong \bigcup_\alpha U_\alpha \times \mathbb{R}^n,$$

so

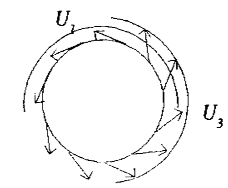
$$\mathcal{E} = \pi^{-1}(M) \cong M \times \mathbb{R}^n.$$

Corollary.

(1) *Every* <u>*oriented*</u> *real line bundle over a 1-manifold is trivial. (ie* $\dim M = 1$ *iff* $M \cong S^1$ *or* $\mathbb{R}^1$ *or* $[a, b)$ *for some* $a < b$*. Then for every such line bundle* $\mathcal{E}$ *one has* $\mathcal{E} \cong M \times \mathbb{R}^1$*.)*

(2) *Every complex line bundle over a 1-manifold is trivial.*

Proof. Exercise.



Example. In the § 1 it is explained how to trivialize TS^1 using the vector field

$$U(x, y) = -y\frac{\partial}{\partial u_1} + x\frac{\partial}{\partial u_2}.$$

On U_1, $U(x, y) = -\sqrt{1 - x^2}\frac{\partial}{\partial x}$;
On U_3, $U(x, y) = \sqrt{1 - y^2}\frac{\partial}{\partial y}$.
So the two trivializations on U_1 and U_3 are given by the functions

$$f_1(x, y) = -\sqrt{1 - x^2}, \qquad f_3(x, y) = \sqrt{1 - y^2}$$
$$= -y \qquad\qquad\qquad = x$$

and so the transition function is

$$g_{13} = \frac{-y}{x} = \lambda_2 \mathrm{id} \lambda_3^{-1}.$$

and so in this case we take

$$\lambda_1(x, y) = -y \qquad \lambda_3(x, y) = x.$$

Let's look a bit more closely at the <u>local</u> meaning of a gauge transformation. It is given by a map (C^∞) in each bundle chart

$$g_\alpha : U_\alpha \longrightarrow GL(n;\mathbb{R}).$$

Its effect is to change the basis or 'frame' in each fibre over U_2. To see that, first let us note the following.

Proposition. *A local trivialization*

$$\psi : \pi^{-1}(U) \cong U_\alpha \times \mathbb{R}^n$$

is equivalent to giving a gauge for $\mathcal{E}_{|_U}$ *—that is, n non-zero and linearly independent sections* $S_i : U \longrightarrow \mathcal{E}_{|_U} = \pi^{-1}(U)$.

Proof. Given n such sections define $\psi(v) = (m, \alpha_1(m), \dots, \alpha_n(m))$ where

$$v = \alpha_1(m)S_1(m) + \cdots + \alpha_n(m)S_n(m).$$

Conversely, given ψ, define $S_i(m) = \psi^{-1}(\{m\} \times e_i)$, where

$$\{e_1, \dots, e_n\}$$

is the canonical basis for $\mathbb{R}^n$. □

So over each U_α we have a moving frame

$$m \longmapsto \{s_1(m), \dots, s_n(m)\}$$

—basis for $\pi^{-1}(m) = \mathcal{E}_m$. It is 'moving' in the sense that it depends on $m \in M$, ie it may not necessarily be constant.

That it is a crucial aspect in geometry and physics —it reflects the way the constant frame is being twisted by an external gauge field; e.g. it accounts for the magnetic field in electromagnetism even though the bundle in that case is trivial.

The effect of the gauge transformation is to change the frame

$$f(m) = \{e_1(m), \dots, e_n(m)\}$$

to

$$(*) \qquad \underset{= f(m)g(m)}{(fg)(m)} = \left(\sum_{i=1}^{n} g_{i1}(m)e_i(m), \dots, \sum_{i=1}^{m} g_{in}(m)e_i(m) \right).$$

A GAUGE TRANSFORMATION IS (LOCALLY) A CHANGE OF FRAME.

So because we are interested in the effect gauge transformations have on objects associated to a bundle (since they represent conserved quantities), we must analyse in local terms (local formulas) the precise effect of such transformations. So perhaps the first object to look at are sections (fermion fields) —one has:

Proposition. *Let* $s \in C^\infty(M, \mathcal{E})$ *and* $s(f)$ *denote* $s : U \longrightarrow \mathcal{E}_{|_U}$ *written with respect to the frame f, and let* $s(fg)$ *denote s written with respect to the transformed frame. Then*

$$\boxed{s(fg) = g^{-1}s(f)}$$

Given a local frame $f = \{e_1, \dots, e_n\}$ of $\mathcal{E}$ over U we define the matrix of the metric by

$$h_{ij}(m) = \langle e_i(m), e_j(m)\rangle_m.$$

Because $\langle , \rangle_m$ is an inner-product then $(h_{ij}(m))$ is a positive definite symmetric matrix, and is the local representative of the metric $\langle , \rangle$ with respect to the local frame.

Given $s_1(m) = \sum_{i=1}^n \alpha_i(m)e_i(m)$, $s_2(m) = \sum_{i=1}^n \beta_i(m)e_i(m)$, $s_1, s_2 \in C^\infty(U, \mathcal{E})$ we have

$$\langle s_1(m), s_2(m)\rangle_m = \left\langle \sum_i \alpha_i(m)e_i(m), \sum_i \beta_i(m)e_i(m) \right\rangle$$
$$= \sum_{i,j} \alpha_i(m)h_{ij}(m)\beta_j(m) \qquad \text{by linearity of } \langle , \rangle$$

that is

$$\langle s_1(m), s_2(m)\rangle_m = s_1(f)(m)^T h s_2(f)(m) \qquad s_1(f) = (\alpha_1, \dots, \alpha_n)$$
$$s_1^T h s_2. \qquad s_2(f) = (\beta_1, \dots, \beta_n).$$

It may also be the case that the gauge group of the bundle is effectively a subgroup G of $GL(n;\mathbb{R})$; more precisely, we say that the (structure or) gauge group can be reduced to G if it possible to choose the local gauges

$$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$$

so that the transition functions

$$g_{\alpha\beta}(m) \in G \subset GL(n;\mathbb{R}).$$

Equivalently, that means given $g'_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n;\mathbb{R})$ it is possible to find a gauge transformation $g \in \mathcal{G}(\mathcal{E})$ given locally by

$$g_\alpha : U_\alpha \longrightarrow GL(n;\mathbb{R})$$

with

$$g'_{\alpha\beta} = g_\alpha g_{\alpha\beta} {g_\beta}^{-1}$$

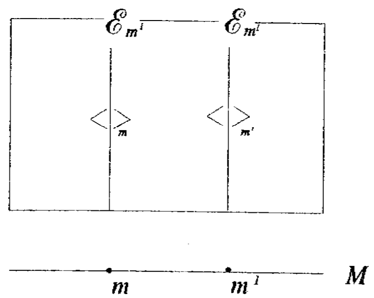
and $g_{\alpha\beta} \in G$. As an example, we note the

Proposition. *If* $\mathcal{E}$ *is orientable then the structure group can allways be reduced to* $SO(n) \subset GL(n;\mathbb{R})$. *Such a reduction is equivalent to defining a metric on* $\mathcal{E}$. *(*$SO(n) = \{ A \in GL(n;\mathbb{R}) : A^T A = 1 \}$.*)*

Definition. A metric on $\mathcal{E}$ is an assignement of an inner-product $\langle , \rangle_m$ to each fibre $\mathcal{E}_m = \pi^{-1}(m)$ such that for any open set $U \subset M$ and $s_1, s_2 \in C^\infty(U, \mathcal{E})$ the function

$$\langle s_1, s_2 \rangle : U \to \mathbb{R}$$

given by



$$\langle s_1, s_2\rangle(m) = \langle s_1(m), s_2(m)\rangle_m$$

is C^∞.

Because this is a new object in our bundle technology we must see how it transforms under a local gauge transformation. I leave it as an exercise to show

$$\boxed{h(fg) = g^T h(f) g}$$

where $h(f)$ denotes the matrix of the metric with respect to the frame f.

Proof of Proposition. Suppose the structure group can be reduced to $SO(n)$. Then

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(n).$$

Over $\pi^{-1}(U_\alpha) \to U_\alpha$ with a local trivialization $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ define

$$\langle \xi(m), \eta(m)\rangle_m = \big\langle \psi_\alpha(\xi(m)), \psi_\alpha(\eta(m))\big\rangle_{\text{Eucl.}} = \psi_\alpha(\xi)^T \psi_\alpha(\eta).$$

where $\langle , \rangle_{\text{Eucl.}}$ denotes the standard Euclidean metric on $\mathbb{R}^n$.

To see that is a good definition we must show it does not change over $U_\alpha \cap U_\beta$ when we change trivialization, ie for $m \in U_\alpha \cap U_\beta$ we have

$$\langle \xi(m), \eta(m)\rangle_m = \big\langle \psi_\beta(\xi(m)), \psi_\beta(\eta(m))\big\rangle_{\text{Eucl.}}$$

as well. But $\psi_\alpha = g_{\alpha\beta}\psi_\beta$ and so

$$\begin{aligned}\big\langle \psi_\alpha(\xi(m)), \psi_\alpha(\eta(m))\big\rangle_{\text{Eucl.}} &= \big\langle g_{\alpha\beta}\psi_\beta(\xi(m)), g_{\alpha\beta}\psi_\beta(\eta(m))\big\rangle_{\text{Eucl.}} \\ &= \big\langle \psi_\beta(\xi(m)), \psi_\beta(\eta(m))\big\rangle_{\text{Eucl.}}\end{aligned}$$

—by def$^{\underline{n}}$ of $SO(n)$. Hence $\langle , \rangle$ is well-defined.

Conversely, if $\mathcal{E}$ has a metric then we may choose just those

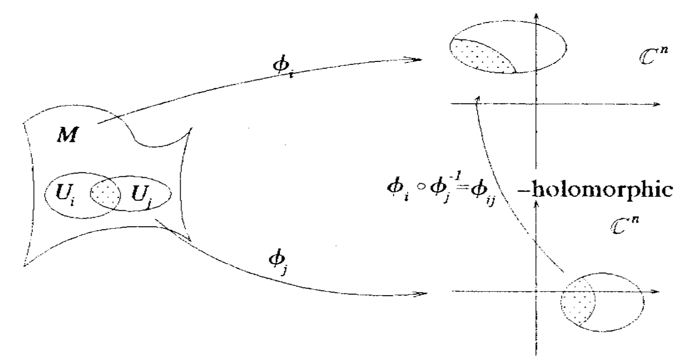
$$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$$

that take frames of $\pi^{-1}(U_\alpha)$ orthonormal with respect to $\langle , \rangle$, to frames orthonormal with respect to $\langle , \rangle_{\text{Eucl.}}$ —this is always possible by the Gram-Schmidt process. That means the transition function will map between frames o.n. for $\langle , \rangle_{\text{Eucl.}}$ and hence they lie in $O(n)$. Because $\mathcal{E}$ is orientable they have positive determinant and hence are in $SO(n)$. □

In particular, if we consider $TM = \mathcal{E}$ then a metric on $\mathcal{E}$ is called a Riemannian metric —and we now formally announce our arrival in the realm of differential geometry. Before discussing that more let us see some examples of vector bundles—.

It will be useful to extend our definition of manifold to a slightly wider category.

Definition. A topological space M is a complex n-manifold if M is locally homeomorphic to $\mathbb{C}^n$ and the transition functions are holomorphic.



The essential motivation is that on a complex manifold one can talk about holomorphic functions $f : M \to \mathbb{C}$, just as on a C^∞-manifold one can talk about smooth functions (or real-analytic functions).

Example. $\mathbb{C}^n$ with a single chart $(\mathbb{C}^n, \text{id})$.

Example. The 2-sphere S^2 with atlas consisting of 2-charts defined by stereographic projection.

(More generally any compact 2-surface can be given the structure of a complex manifold).

Example. Complex projective space and the complex Grassmanian. Complex projective space is defined by

$$\boxed{\mathbb{C}P^n = \{\text{one-dimensional (complex) subspaces of } \mathbb{C}^{n+1}\}}$$

(Note that $\mathbb{C}P^1 \cong S^2$). So a point $[l] \in \mathbb{C}P^n$ parameterizes a line in $\mathbb{C}^{n+1}$ passing thru'the origin. We define an atlas on $\mathbb{C}P^n$ in the following way. The topology on $\mathbb{C}P^n$ is the quotient space topology comming from the quotient map

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}.$$

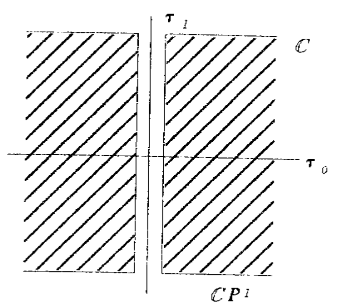
For $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ we let

$$\pi(z) = [z_0, \dots, z_n].$$

One then calls $(z_0, \dots, z_n)$ homogeneous coordinates of $[z_0, \dots, z_n]$ —because if $z_i' = \lambda z_i$, $\lambda \in \mathbb{C}$ then $(z_0', \dots, z_n')$ are also homogeneous coordinates. An atlas is defined on $\mathbb{C}P^n$ by $n+1$ charts

$$U_i = \{\, \omega \in \mathbb{C}P^n : \omega = [z_0, \dots, z_n] \text{ and } z_i \neq 0 \,\}$$

$$\phi_i : U_i \longrightarrow \mathbb{C}^n, \qquad \phi_i(\omega) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i} \right).$$



(Note independent of choice of homog. coords.). It is easy to see that ϕ_i is a homeomorphism and $\phi_{ij} = \phi_i \circ \phi_j{}^{-1}$ is a diffeomorphism $\mathbb{C}^n \to \mathbb{C}^n$.

An important generalization of complex projective space are the Grassmanian manifolds

$$\boxed{\mathrm{Gr}_k(\mathbb{C}^n) = \{k\text{-dimensional (complex) subspaces of } \mathbb{C}^n\}}.$$

So, in particular, $\mathrm{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{C}P^n$. The manifold structure is defined in an analogous way to $\mathbb{C}P^n$.

Exercises.

(1) Prove that $\mathbb{C}P^1 \cong S^2$ (analogous to $\mathbb{R}P^1 \cong S^1$).
(2) Prove that $\mathbb{C}P^{n-1} \cong S^{2n-1}/U(1)$.

If M is a complex manifold then a vector function $f : M \to \mathbb{C}^n$ is said to be holomorphic if each of the compositions

$$f \circ \phi_i{}^{-1} : V_i \subset \mathbb{C}^m \longrightarrow \mathbb{C}^n \qquad V_i = \phi_i(U_i)$$

are holomorphic, where $\mathcal{A}_M = \{\, (U_i, \phi_i) : i \in \Lambda \,\}$ is a complex atlas for M. We denote the space of such functions by $\Gamma_{\text{hol}}(M; \mathbb{C}^n)$.

If M, N are complex manifolds with atlases $\mathcal{A}_M = \{(U_i, \phi_i)\}$ and $\mathcal{A}_N = \{(W_j, z_j)\}$, then a C^∞ map $f : M \to N$ is said to be holomorphic if each of the composite maps $\mu_{ji} = \tau_j \circ f \circ \phi_i{}^{-1} : \mathbb{C}^m \to \mathbb{C}^n$:

$$\begin{array}{ccc} \mathbb{C}^m & \longrightarrow & \mathbb{C}^n \\ \big\downarrow{\scriptstyle \phi_i{}^{-1}} & & \big\uparrow{\scriptstyle \tau_j} \\ M & \xrightarrow{\ f\ } & N \end{array}$$

is holomorphic in the usual sense.

Definition. Let $\mathcal{E} \xrightarrow{\pi} M$ be a complex vector bundle where M is a complex manifold. Then if the transition functions

$$g_{ij}(m) : \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad g_{ij} : U_i \cap U_j \longrightarrow GL(n; \mathbb{C})$$

are holomorphic maps (and the projection π is a holomorphic map) then $\mathcal{E}$ is said to be a holomorphic vector bundle.

If $\pi : \mathcal{E} \to M$ is a holomorphic vector bundle then

$$\Gamma_{\text{hol}}(M; \mathcal{E}) = \{\text{hol. maps } s : M \to \mathcal{E} \text{ with } \pi \circ s = \text{id}\}$$

is called the space of holomorphic sections of $\mathcal{E}$. (If M is a compact manifold then by generalized Riemann-Roch Theorem $\dim \Gamma_{\text{hol}}(M; \mathcal{E})$ is finite.)

Exercise. Locally a holomorphic section of $\mathcal{E}$ is given by holomorphic maps

$$f_i : U_i \longrightarrow \mathbb{C}^n$$

with $f_j(x) = g_{ij}(x) f_i(x)$. Thus the idea is just as we introduce C^∞-real vector bundles to talk about 'twisted' smooth functions, we introduce holomorphic bundles to talk about 'twisted' holomorphic functions.

Examples of Vector Bundles.

(1). The trivial bundle $\mathcal{E} \to M$ given by $\mathcal{E}_{\text{triv}} = M \times \mathbb{R}^n$. Similarly one has the trivial complex bundle $\mathcal{E}^{\mathbb{C}}_{\text{triv}} = M \times \mathbb{C}^n$. In this case

$$C^\infty(M; \mathcal{E}_{\text{triv}}) \cong C^\infty(M, \mathbb{R}^n) \qquad (m \longmapsto \{m\} \times \underset{f(m)}{\underset{=}{\{v\}}})$$

$$C^\infty(M; \mathcal{E}^{\mathbb{C}}_{\text{triv}}) \cong C(M; \mathbb{C}^n).$$

If M is a complex manifold then

$$\Gamma_{\text{hol}}(M; \mathcal{E}^{\mathbb{C}}_{\text{triv}}) \cong \Gamma_{\text{hol}}(M; \mathbb{C}^n).$$

(2). <u>The tangent bundle</u>
If M is a C^∞ manifold then we have seen explicitly that TM is a C^∞ vector bundle over M.

$\mathbb{C}^n$ is a complex manifold and has associated to each point $x \in \mathbb{C}^n$ the holomorphic tangent space (of complex dimension n)

$$T_x\mathbb{C}^n = \{\text{holomorphic linear derivations}\}$$

So $T_x\mathbb{C}^n$ has the canonical basis

$$\left\{ \frac{\partial}{\partial z_1}\bigg|_x, \dots, \frac{\partial}{\partial z_n}\bigg|_x \right\} \qquad \frac{\partial}{\partial z_i}\bigg|_x = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)\bigg|_x .$$

There is an obvious vector space isomorphism

$$T^{\mathbb{C}}_m M \cong T_{\phi_i(m)}\mathbb{C}^n$$

where $T^{\mathbb{C}}_m M = \{\text{holomorphic linear derivations on } M\}$, defined by a local chart (U_i, ϕ_i) around $m \in M$. The space

$$T^{\mathbb{C}}M = \bigcup_{m\in M} T^{\mathbb{C}}_m M$$

is called the holomorphic tangent bundle to M. The bundle structure arises in a completely similar way to that of the real tangent bundle. Clearly,

$$C^\infty(M; T^{\mathbb{C}}M) = \text{smooth complex vector fields on } M$$

and

$$\Gamma_{\text{hol}}(M; T^{\mathbb{C}}M) = \text{holomorphic vector fields on } M$$

—locally of the form $X(m) = \sum \alpha_i(m)\frac{\partial}{\partial z_i})$ where $\alpha : U \subset M \to \mathbb{C}^n$ are holomorphic functions and z_i are local coordinates in U.

Let $\mathcal{E}$, $\mathcal{E}'$ be vector bundles over M with transition functions g_{ij}, g'_{ij}. Then

(3) (Direct sum). $\mathcal{E} \oplus \mathcal{E}'$ is a vector bundle over M with transition functions $\begin{pmatrix} g_{ij} & 0 \\ 0 & g'_{ij} \end{pmatrix} = g_{ij} \oplus g'_{ij}$ $(\mathcal{E} \oplus \mathcal{E}')_m = \mathcal{E}_m \oplus \mathcal{E}'_m$.

(4) (Tensor product). $\mathcal{E} \otimes \mathcal{E}'$ is a vector bundle over M with transition functions $g_{ij} \otimes g'_{ij}$.

(5) (The dual bundle). $\mathcal{E}^*$ is a v. bundle with transition functions $(g_{ij}{}^{-1})$.

Corollary. *(*$\text{Hom}_m = \text{Hom}(\mathcal{E}_m, \mathcal{E}'_m)$ *is a v. space of dim.* n^2 *if* $\dim \mathcal{E} = n = \dim \mathcal{E}'$*)* $\text{Hom}(\mathcal{E}, \mathcal{E}')$ *is a vector bundle because* $\text{Hom}(V, W) \cong V^* \otimes W$ *for v. spaces* V *and* W.

(6) Exterior bundle. $\bigwedge^k \mathcal{E}$ is a vector bundle with transition functions $g_{i1} \wedge \cdots \wedge g_{ij}$.

(7) Tautological bundle. We consider the complex Grassmanian $\text{Gr}_k(\mathbb{C}^m)$. This is a complex manifold and over $\text{Gr}_k(\mathbb{C}^m)$ there is a very natural complex vector bundle $\mathcal{E} \to \text{Gr}_k(\mathbb{C}^m)$ with fibre at $[w] \in \text{Gr}_k(\mathbb{C}^m)$ the subspace $W \subset \mathbb{C}^m$ parameterized by that point.

The dual of the determinant bundle $\text{Det}(\mathcal{E}^*)$ is a holomorphic line bundle over $\text{Gr}_k(\mathbb{C}^n)$ and it is not hard to show that

$$(**) \qquad \Gamma_{\text{hol}}(\text{Gr}_k(\mathbb{C}^p); \text{Det}(\mathcal{E}^*)) \cong \bigwedge^k (\mathbb{C}^p)^*.$$

(The determinant bundle $\text{Det}(V)$ associated to a vector bundle $\pi : V \to M$ is the line bundle whose fibre at $n \in M$ is $\bigwedge^{\max} V_m = \bigwedge^{\max} \pi^{-1}(m)$.)

Let us see that for the case $k = 1$, $m = n+1$, ie for complex projective space $\mathbb{C}P^n$. Notice in this case that $\text{Det}(\mathcal{E}^*) = \mathcal{E}^*$.

First observe any $v \in \mathcal{E}$ can be written

$$v = \lambda(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$$

where $(z_0, \dots, z_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, and the projection $\pi : \mathcal{E} \to \mathbb{C}P^n$ is

$$\pi(v) = [z_0, \dots, z_n] \in \mathbb{C}P^n.$$

Suppose $v \in \pi^{-1}(U_\alpha)$, then ($U_\alpha = \{[z_0, \dots, z_n] \in \mathbb{C}P^n : z_\alpha \neq 0\}$)

$$v = \lambda_\alpha \left(\frac{z_0}{z_\alpha}, \dots, \underbrace{1}_{\alpha}, \dots, \frac{z_n}{z_\alpha} \right) \qquad \lambda_\alpha = \lambda z_\alpha.$$

Define

$$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^1, \qquad \psi_i(v) = ([z_0, \dots, z_n], \quad \lambda_\alpha \quad)$$

note this is unique

Then ψ_i is a diffeomorphism (CHECK!) and $\mathbb{C}$-linear.

Suppose $v \in \pi^{-1}(U_\alpha \cap U_\beta)$, then we have

$$\psi_\alpha(v) = ([z_0, \dots, z_n], \lambda_\alpha) \qquad \psi_\beta(v) = ([z_0, \dots, z_n], \lambda_\beta).$$

But by definition

$$\lambda_\alpha = \frac{z_\alpha}{z_\beta}\lambda_\beta.$$

So we define transition function

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(1;\mathbb{C}) = \mathbb{C}^*$$

by

$$g_{\alpha\beta}([z]) = \frac{z_\alpha}{z_\beta} \qquad [z] = [z_0, \ldots, z_n]$$

Hence $g_{\alpha\beta}([z])$ is holomorphic and

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \text{identity} \qquad g_{\alpha\alpha} = \text{identity}.$$

Hence $\mathrm{Det}(\mathcal{E}) = \mathcal{E}$ and $\mathcal{E}^*$ are holomorphic line bundles over $\mathbb{C}P^n$. Let's try and deduce the sections. In fact, we can do this for the more general situation of $\mathcal{L}^{\otimes m} \longrightarrow \mathbb{C}P^n$ where $\mathcal{L} = \mathcal{E}^*$ and m is an integer. Because holomorphic line bundles are parameterized by the sheaf cohomology group $H^2(\mathbb{C}P^n; \mathcal{C}^*)$ where $\mathcal{C}^*$ is the sheaf of non-zero holomorphic functions on $\mathbb{C}^{n+1}$, and because the Hodge descomposition theorem identifies an isomorphism

$$C_1 : H^2(\mathbb{C}P^n; \mathcal{C}^*) \longrightarrow \mathbb{Z}$$

where C_1 assigns to a line bundle its Chern class, then we may deduce that these are all of the holomorphic line bundles over $\mathbb{C}P^n$, up to isomorphism (I realize I have used some ideas here that have not been introduced in the text.)

To see what a section of $\mathcal{L}^{\otimes m}$ is notice that such a section is equivalent to giving compatible holomorphic functions f_α in each U_α. Because the $[z_1, \ldots, z_n]$ are homogeneous coordinates we require

$$(1) \qquad f_\alpha(\lambda z_1, \ldots, \lambda z_n) = f_\alpha(z_1, \ldots, z_n).$$

On the intersection $U_\alpha \cap U_\beta$ we require

$$(2) \qquad f_\alpha = (g'_{\alpha\beta})^m f_\beta = \left(\frac{z_\beta}{z_\alpha}\right)^m f_\beta.$$

So $f = z_\alpha^m f_\alpha = z_\beta^m f_\beta$ is independent of the labels $\alpha, \beta, \ldots$. Further, the homogeneity condition (1) is replaced by

$$f(\lambda z_1, \ldots, \lambda z_n) = \lambda^m f(z_1, \ldots, z_n).$$

Hence, since f is holomorphic, we deduce

$$\Gamma_{\mathrm{hol}}(\mathbb{C}P^n; \mathcal{L}^{\otimes m}) = \begin{cases} \left\{ \begin{array}{l} \text{space of homogeneous} \\ \text{polynomials of degree } m \\ \text{in } z_1, \ldots, z_n. \end{array} \right\} & m \geq 0 \\ \emptyset & m < 0. \end{cases}$$

Notice in particular that for $m = 1$, we have

$$\Gamma_{\mathrm{hol}}(\mathbb{C}P^n; L) = \{\text{space of linear forms on } \mathbb{C}\}$$

which is precisely (**) for the case $k = 1$, $m = p + 1$.

Notice further one can hence calculate by purely combinatorial arguments that

$$\dim \Gamma_{\mathrm{hol}}(\mathbb{C}P^p; \mathcal{L}^{\otimes m}) = \binom{m + p - 1}{p}.$$

This can also be calculated topologically using the Riemann-Roch-Hirabruch theorem.

3. Differential Forms, Maxwell's Equations and 0 + 1-Dimensional TQFT

Let M be a C^∞ manifold. Then a differential k-form ω on M is an element of

$$\boxed{\Omega^k(M) \overset{\text{def}^{\text{n}}}{=} C^\infty(M; \textstyle\bigwedge^k T^*M)}$$

So at each $m \in M$ we have

$$\omega(m) \in \bigwedge^k T_m^*M.$$

T_mM has a canonical basis $\left\{ \frac{\partial}{\partial x_1}\Big|_m, \ldots, \frac{\partial}{\partial x_n}\Big|_m \right\}$ rel. to local coordinates $(x_1, \ldots, x_n)$ near m and hence T_m^*M has a dual basis

$$\{dx_1|_m, \ldots, dx_n|_m\}$$

defined by

$$\boxed{dx_i\left(\frac{\partial}{\partial x_j}\right)\Big|_m = \delta_{ij}}$$

So we may write

$$\omega(m) = \sum_I f_I dx_I$$

$I = (i_1, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

$$f_I = f_{i_1 i_2 \ldots i_k} \qquad dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

$$\text{The } dx_i \text{ satisfy } \begin{cases} dx_i{}^2 = 0 \\ dx_i \wedge dx_j = -dx_j \wedge dx_i. \end{cases}$$

So $\omega(m)$ is a multilinear alternating map

$$\omega(m) : \underbrace{T_mM \otimes \cdots \otimes T_mM}_{k} \longrightarrow \mathbb{R}$$

$$\omega_m(v_{p(1)}, \ldots, v_{p(k)}) = \mathrm{sgn}(p)\omega_m(v_1, \ldots, v_k)$$

for p a permutation on k 'letters'.

There is a canonical operator

$$\begin{array}{rcl} d : \Omega^k(M) & \longrightarrow & \Omega^{k+1}(M) \\ \omega & \longmapsto & d\omega \end{array}$$

defined locally by

$$d\omega = d\left(\sum f_I dx_I\right) = \sum df_I \wedge dx_I$$

Note $f \in C^\infty(M) = \Omega^0(M)$ so $df_I \in \Omega^1(M)$ and locally

$$df_I = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

If $X \in C^\infty(M;TM)$, $X = \sum \alpha_i \frac{\partial}{\partial x_i}$, then locally

$$\begin{aligned} df(X) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \left(\sum \alpha_j \frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j=1}^{n} \alpha_j \frac{\partial f}{\partial x_i} dx_i \left(\frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j=1}^{n} \alpha_j \frac{\partial f}{\partial x_i} \delta_{ij} \\ &= \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}. \end{aligned}$$

So

$$\boxed{df(X) = X(f)}$$

Example. $M = \mathbb{R}^3$.

$$\underset{\dim 1}{\Omega^0(\mathbb{R}^3)},\ \underset{\dim 3}{\Omega^1(\mathbb{R}^3)},\ \underset{\dim 3}{\Omega^2(\mathbb{R}^3)},\ \underset{\dim 1}{\Omega^3(\mathbb{R}^3)}$$

$(\Omega^k(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3) \otimes \Omega^k)$

$$\begin{array}{ccccc} 0 & \longrightarrow & \Omega^0(\mathbb{R}^3) & \xrightarrow[(1)]{d} & \Omega^1(\mathbb{R}^3) \\ & & & & \downarrow d\ (2) \\ 0 & \longleftarrow & \Omega^3(\mathbb{R}^3) & \xleftarrow[(3)]{d} & \Omega^2(\mathbb{R}^3) \end{array}$$

(1) On functions: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

(2) On 1-forms: $d(f_1 dx + f_2 dy + f_3 dz)$
$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) dx \wedge dz$
$+ \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy$

(3) On 2-forms: $d(f_1 dy \wedge dz - f_2 dx \wedge dz + f_3 dx \wedge dy)$
$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx \wedge dy \wedge dz$

Hence

d(0-forms) = gradient $\nabla \underline{f}$ d(1-forms) = curl $\nabla \times \underline{f}$ d(2-forms) = divergence $\nabla \cdot \underline{f}$	$\underline{f} = (f_1, f_2, f_3)$

The exterior derivate operator has the following properties:
If $\omega_1 = \sum f_I dx_I$ (locally), $\omega_2 = \sum g_J dx_J$ (locally)

(1) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1}\omega_1 \wedge d\omega_2$ (anti-derivation).
Locally: $\omega_1 \wedge \omega_2 = \sum f_I g_J dx_I \wedge dx_J$, (1) $\iff$ $d(f_I g_J) = df_I \cdot g_J + f_I \cdot dg_J$.

(2) $d^2 = 0$, ie $d(d\omega) = 0$, $\omega \in \Omega^k(M)$.
Locally: $(f \in C^\infty(M))$, (2) $\iff$ $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (Flat connection!)

For a C^∞ manifold M of dim n there is the de Rham complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & & \\ & & & & \downarrow d & & \\ 0 & \longleftarrow & \Omega^n(M) & \xleftarrow{d} & \dots & \xleftarrow{d} & \end{array}$$

and hence the de Rham cohomology groups

$$\begin{aligned} H^k(M) &= \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\} \\ &= \operatorname{Ker}\{d : \Omega^k \to \Omega^{k+1}\}/\operatorname{Im}\{d : \Omega^{k-1} \to \Omega^k\}. \end{aligned}$$

Thus $H^k(M)$ measures the failure of the de Rham complex to be an exact sequence. We say that

- $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$.
- $\omega \in \Omega^k(M)$ is exact if $\omega = d\tau$, some $\tau \in \Omega^{k-1}$.

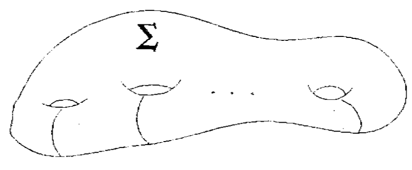
$H^k(M)$ is a topological invariant of M.
In particular,

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases} \qquad \text{(ie } \mathbb{R}^n \text{ has no topology).}$$

$\implies \omega \in \Omega^k(\mathbb{R}^n)$ and $d\omega = 0$ then there exists: $\tau \in \Omega^{k-1}(\mathbb{R}^n)$, $d\tau = \omega$. (ie τ exists globally)

Locally every manifold $\cong \mathbb{R}^n$. So given $\omega \in \Omega^k(\mathbb{R}^n)$ with $d\omega = 0 \implies \omega = d\tau$ some $\tau \in \Omega^{k-1}(M)$ LOCALLY (Poincar lemma). But in general τ does not necessarily exist globally —$H^k(M)$ represents "obstruction" to global existence of τ.

An important topological invariant of M is its Euler number defined by $\chi(M) = \sum_{k=1}^{n}(-1)^k \dim H^k(M)$. In particular, when $M = \Sigma_g$ is a compact surface with g holes then $H^0(\Sigma_g) = \mathbb{R}$, $H^1(\Sigma_g) = \mathbb{R}^{2g}$, $H^2(\Sigma_g) = \mathbb{R}^1$ so $\chi(\Sigma_g) = 2 - 2g$, which agrees with the classical Euler number of a surface.



3.1 Maxwell's Equations.

$\nabla \cdot B = 0$	(no magnetic charges)
$\nabla \times E + \frac{\partial B}{\partial t} = 0$	(Faraday law: a changing magnetic field produces an electric field)
$\nabla \cdot E = \rho$	(Gauss' law)
$\nabla \times B - \frac{\partial E}{\partial t} = \underset{\sim}{j}$	(Ampere's law—$\frac{\partial E}{\partial t}$: a changing electric field produces a magnetic field)

$$\underset{\text{electric field}}{E = (E_1, E_2, E_3)}, \qquad \underset{\text{magnetic field}}{B = (B_1, B_2, B_3)}$$

Define the Faraday 2-form F by:

$$\begin{aligned} F &= -\frac{1}{2} F_{ij}\, dx_i \wedge dx_j \\ &= E_1 dx_1 \wedge dt + E_2 dx_2 \wedge dt + E_3 dx_3 \wedge dt \\ &\quad + B_3 dx_1 \wedge dx_2 + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1, \end{aligned}$$

where $(x_1, x_2, x_3, \underset{x_4}{\underset{=}{t}})$ are canonical coords. on $\mathbb{R}^4$.

As a matrix in basis $dx_i \wedge dx_j$

$$F = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}$$

And the $*$-form (defined by Minkowski metric)

$$\begin{aligned} *F &= -\tfrac{1}{2} F_{ij} * (dx_i \wedge dx_j) \\ &= -E_1 dx_2 \wedge dx_3 + E_2 dx_1 \wedge dx_3 - E_3 dx_1 \wedge dx_2 \\ &\quad + B_1 dx_1 \wedge dt + B_2 dx_2 \wedge dt + B_3 dx \wedge dt. \end{aligned}$$

$$* : \Omega^k(\mathbb{R}^4) \longrightarrow \Omega^{4-k}(\mathbb{R}^4)$$
$$*(dx_i \wedge dx_j) = \pm dx_k \wedge dx_l$$
$$(k, l \neq i, j) \quad *^2 = +1$$

The Maxwell's equations become

$$\boxed{dF = 0 \quad d * F = J} \qquad J \text{ current density 3-form,}$$

which is the abelian version of the Yang-Mills equations.

But we know $H^2(\mathbb{R}^4) = 0$ and so there is $A \in \Omega^1(\mathbb{R}^4)$ with

$$(*) \qquad F = dA \qquad \begin{bmatrix} \text{Note we can change } A \text{ by} \\ A \longrightarrow A + d\alpha,\ \alpha \in \Omega^0(\mathbb{R}^4) \end{bmatrix}$$

where in coordinates

$$A = A_i dx_i \qquad \text{(vector potential)}$$

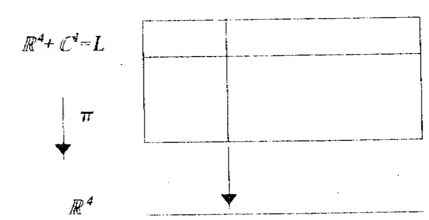
and $(*) \Longrightarrow$

$$F = (F_{ij}) \qquad F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \qquad \begin{pmatrix} \text{ie } d(\text{1-form}) \\ = \text{curl} \end{pmatrix}$$

Another way to express Maxwell's equations is that the field strength (F) is the curvature of a connection

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i \qquad \begin{bmatrix} \text{The connection is only defined up to a} \\ U(1)\text{-gauge (phase) transformation.} \end{bmatrix}$$

on a trivial complex line bundle L over $\mathbb{R}^4$.



3.2 Connections on Vector Bundles.

The theory of connections, or covariant derivates arise from following observation: Consider a vector bundle $\pi : \mathcal{E} \to M$. We may suppose that the dynamics of some physical system aris as sections $s : M \to \mathcal{E}$, $\pi \circ s = \text{id}$. To calculate the "acceleration" of s we must calculate its differential. That gives us a map

$$ds : TM \longrightarrow T\mathcal{E},$$

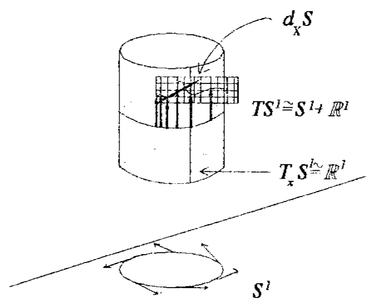
and with respect to a 'direction' $X \in C^\infty(M, TM)$, we obtain a directional derivative

$$d_X s = ds(X) : M \longrightarrow T\mathcal{E}.$$

That is, the derivative of s in the direction X is a map

$$\boxed{d_X : C^\infty(M, \mathcal{E}) \longrightarrow C^\infty(M, T\mathcal{E})}$$

However, we expect the derivative of a section with respect to X to be a section (first as the derivative of a function is a function —corresponding to $\mathcal{E} = M \times \mathbb{C}$). The remedy is to



project $d_X s$ back into fibre.

$$\nabla_X s = p \circ d_X s \qquad p : T\mathcal{E} \longrightarrow V\mathcal{E}$$

$$\underbrace{V\mathcal{E} = \operatorname{Ker} d\pi}$$

Vertical bundle= sub-bundle of $T\mathcal{E}$
of vectors tangent to the fibres $\pi^{-1}(M)$.

Let's see what happens in the simplest case $\mathcal{E}_{\text{triv}} = M \times \mathbb{R}^n$.

$$C^\infty(M; \mathcal{E}_{\text{triv}}) \cong C^\infty(M, \mathbb{R}^n)$$
$$s \longleftrightarrow f$$
$$s(m) = \big(m, f(m)\big).$$

First note that for manifolds M and N there is a natural isomorphism

$$\begin{array}{ccc} T(M \times N) & \cong & TM \oplus TN \\ (v & \longmapsto & d\pi_M(v) \oplus d\pi_N(v)), \end{array}$$

where $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ are the canonical projection maps.

So for $s \in C^\infty(M; \mathcal{E}_{\text{triv}})$ one has

$$d_X s \in T\mathcal{E}_{\text{triv}} = T(M \times \mathbb{R}^n).$$

Define ∇_X by $d\pi_n \circ d_X s$ (ie $P = d\pi_n$) so that

$$\begin{aligned} \nabla_X s &= d\pi_n(d_X s) \in T\mathbb{R}^n \\ &= d(\pi_n \circ s)(X) && \text{(Chain rule)} \\ &= df(X) && s(m) = \big(m, f(m)\big) \\ &= X(f) \end{aligned}$$

—the usual directional derivative.

Let $\pi : M \times \mathbb{R}^n \to M$, $\omega = d\pi_n$ —so $\underset{\text{connection 1-form}}{\omega^1} \in \Omega^1(T\mathcal{E}; V\mathcal{E})$

$$0 \longrightarrow \underset{(=\operatorname{Ker} d\pi_n)}{VE} \underset{\omega}{\overset{\longrightarrow}{\leftarrow}} TE \longrightarrow \underset{(=\operatorname{Ker} d\pi \cong TM)}{\overset{\text{Horizontal sub-bundle}}{\pi^* TM}} \longrightarrow 0$$

A connection on $\mathcal{E}$ is a splitting of this exact sequence —defined by ω.

If we choose $\omega \neq d\pi_n$, then ω and $d\pi_n$ differ by an element of $\operatorname{End}(\mathcal{E})$. So

$$\nabla^\omega_X s = \omega(d_X s) = X(f) + A(\cdot) f$$

or

$$\boxed{\nabla^\omega = d + A} \qquad \text{(At least \underline{locally})}.$$

Just as we characterised derivatives as linear derivations, we characterise covariant derivatives (connections) as linear derivations on sections with image in space of sections:

Definition. A covariant derivative is a map

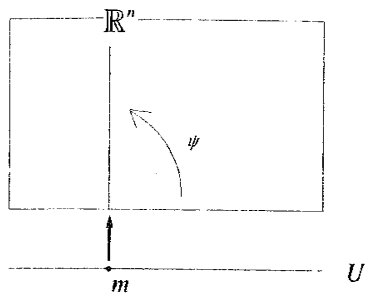
$$\nabla_X : C^\infty(M; \mathcal{E}) \longrightarrow C^\infty(M; \mathcal{E}) \qquad X \in C^\infty(M; TM)$$

such that

$$\nabla_X(fs)(m) = \underset{=X(f)}{d_X f}\, s(m) + f(m) \nabla_X s$$

and

$$\nabla_{fX} s = f \nabla_X s, \qquad \nabla_X(s_1 + s_2) = \nabla_X(s_1) + \nabla_X s_2.$$



LOCALLY

$$\mathcal{E}_{|_U} = \pi^{-1}(U) \overset{\psi}{\cong} U \times \mathbb{R}^n$$

and ψ defines a 'moving' frame $\{e_1, \ldots, e_n\}$,

$$\{e_1(m), \ldots, e_n(m)\} \text{ basis for } \mathcal{E}_m.$$

So for $s \in C^\infty(U; \mathcal{E}_{|_U})$

$$s(m) = \sum_{i=1}^n f_i(m) e_i(m) \qquad f_i : U \longrightarrow \mathbb{R}^1.$$

So that $\nabla e_k = \sum_j \vartheta_{jk} e_j$ for some 1-forms $\vartheta_{jk} \in \Omega^1(U)$. (We refer to $\vartheta_f = (\vartheta_{jk})$ as the connection matrix of ∇)

$$\begin{aligned}
\nabla s &= \nabla\left(\sum_i \xi_i e_i\right) \\
&= \sum_i d\xi_i \otimes e_i + \sum_i \xi_i \otimes \nabla e_i \\
&= \sum_i d\xi_i \otimes e_i + \sum_i \xi_i \otimes \vartheta_{ij} e_i \\
&= \sum_i \left[d\xi_i + \sum_k \vartheta_{ik}\right] \otimes e_i
\end{aligned}$$

We could write $\vartheta_{jk} = \sum_{r=1}^n \Gamma^r_{jk} dx_r$ locally. The symbols $\Gamma^r_{jk} \in C^\infty(U)$ are called Christoffel symbols. In particular for a Riemannian manifold (M, g) there is a canonical construction of the Γ^r_{jk} (Fundamental theorem of Riemannian geometry). Anyway in the

$$\{e_1, \ldots, e_n\} = (f)$$

frame we have

$$\boxed{\nabla s_f = d\xi_f + \vartheta \xi_f = [d + \vartheta_f](\xi_f)}$$

The <u>curvature</u> of the connection is a bundle endomorphism given for any $X, Y \in C^\infty(M; \mathcal{E})$ by

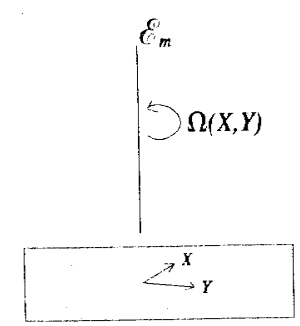
$$\Omega(X,Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s \qquad [A,B] = AB - BA$$

Locally

$$\begin{aligned}
\Omega s_f &= (d + \vartheta_f)(d + \vartheta_f)s_f \\
&= [d\vartheta_f + \vartheta_f \wedge \vartheta_f]s_f.
\end{aligned}$$

Globally,

$$\Omega \in C^\infty(M; \mathrm{End}(\mathcal{E}) \otimes \bigwedge^2 T^*M) = \Omega^2(M; \mathrm{End}(\mathcal{E})).$$



To get a better idea of meaning of Ω consider effect of a gauge transformation $g \in \mathcal{G}(\mathcal{E}) = C^\infty(M; \mathrm{Aut}\,\mathcal{E})$, given locally by maps

$$g_\alpha : U_\alpha \longrightarrow G$$

Changes frame $f = \{e_1, \ldots, e_n\}$ to

$$fg = \left\{\sum_{i=1}^n g_{i1} e_i, \ldots, \sum_{i=1}^n g_{in} e_n\right\} = \{e'_1, \ldots, e'_n\}$$

So have new local form for ∇ given by

$$\boxed{\nabla s_{fg} = d\xi_{fg} + \vartheta_{fg}\xi_{fg}} \qquad s = \sum_i \xi^i_{fg} e'_i$$

But ϑ_{fg} defined by

$$\nabla e'_k = \sum_j \vartheta_{jk}(fg)e'_j$$

(1)
$$= \sum_{i,j} \vartheta_{jk}(fg) g_{ij} e_i$$

also

(2)
$$\nabla\left(\sum_i g_{ij}e_i\right) = \sum_i dg_{ij}e_i + \sum_{i,r} g_{ij}\vartheta_{ri}(f)e_r$$

(1) = (2) $\implies$

$$g\vartheta(fg) = dg + \vartheta(f)g$$

or

(3)
$$\boxed{\vartheta(fg) = g^{-1}dg + g^{-1}\vartheta(f)g}$$

More invariantly the group $\mathcal{G}(\mathcal{E})$ of gauge transformations of $\mathcal{E}$ act on the space of connections $\mathcal{A}$ on $\mathcal{E}$ by

$$\mathcal{G}(\mathcal{E}) \times \mathcal{A} \longrightarrow \mathcal{A}, \qquad (g, \nabla) \longmapsto g^{-1}\nabla g = g(\nabla).$$

This means $g(\nabla)(s) = g^{-1}\nabla(gs)$ for $s \in C^\infty(M;\mathcal{E})$.

A similar calculation shows

(4)
$$\boxed{\Omega(fg) = g^{-1}\Omega(f)g}$$

Note if $\mathcal{L} \to M$ is a complex line bundle then with $U(1)$-gauge groups

$$\text{so}\ \boxed{\begin{array}{c}\vartheta(fg) = id\alpha + \vartheta(f)\\ d + \vartheta(f) \longmapsto d + id\alpha + \vartheta(f)\end{array}}$$

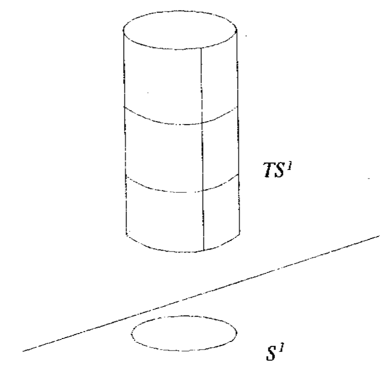
$g = e^{i\alpha}$ $g^{-1}dg = id\alpha$ for some real-valued function $\alpha : U \to \mathbb{R}^1$. Note also

$$\boxed{\Omega(f) = d\vartheta(f)}$$

Now compare this with our earlier discussion of MAXWELL'S EQUATIONS to see (more or less) why electromagnetism is $U(1)$ gauge theory.

Maybe then every connection is gauge equivalent to $\nabla = d$? Of couse... no. Consider $\mathcal{E}_{\text{triv}} = M \times \mathbb{R}^n$, then

$$d_X f = X(f).$$



So $\nabla : C^\infty(M;\mathcal{E}_{\text{triv}}) \to C^\infty(M;\mathcal{E}_{\text{triv}})$ gauge equivalent to d means that $\exists\, g : M \to \mathbb{R}^n$ such that $\nabla^g_X = g^{-1}\nabla_X g = X$, so ($[a,b] = ab - ba$)

$$g^{-1}[\nabla_X, \nabla_Y]g = [\nabla^g_X, \nabla^g_Y] = [X,Y] = \nabla^g_{[X,Y]} = g^{-1}\nabla[X,Y]g.$$

So the obstruction to gauge transforming ∇ to the trivial connection d is measured by

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \Omega(X,Y).$$

Example. In dim 1, $M = \mathbb{R}^1$, $n = 1$ (line bundle). $\mathcal{A}$ = space of unitary connections, $\mathcal{G}(\mathcal{E})$ = gauge group, then the moduli space $\mathcal{M} \stackrel{\text{defn}}{=} \mathcal{A}/\mathcal{G}(\mathcal{E})$ is equal to a point (every connection gauge equivalent to d). To see that, notice every line bundle over $\mathbb{R}^1$ is trivial (why?) and hence a connection ∇ can be written globally as

$$\nabla = \frac{d}{dx} + \varrho(x) = g(x)^{-1}\frac{d}{dx}g(x), \qquad \text{where } g(x) = e^{-\int^x \varrho(t)\,dt}$$

Similarly, $\mathcal{M}$ = pt for any complex n-bundle $\mathcal{E} \to \mathbb{R}^1$ with Hermitian metric.

Suppose $M = \mathbb{R}^n$, then a connection ∇ has components $\nabla_i = \frac{\partial}{\partial x_i} + A_i(x)$ —the question we must answer is does there exist g with $g\nabla_i g^{-1} = \frac{\partial}{\partial x_i}$?

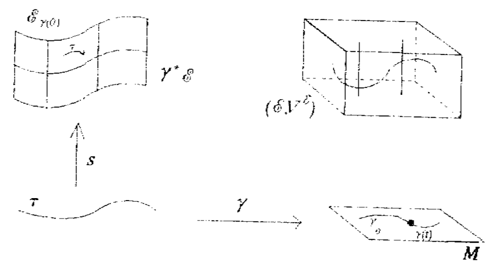
No, because of commutator $[\nabla_i, \nabla_j] = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] = F_{ij}$. (Note $\nabla_{[\partial_i, \partial_j]} = 0$) So $\mathcal{M}$ is non-trivial.

One way to explain $\mathcal{M}$ = pt in dim 1 is by parallel transport. Let $\gamma : \mathbb{R}^1 \to M$ be C^∞ curve

If $s : \mathbb{R}^1 \to \gamma^*\mathcal{E}$ smooth define $\nabla^{(\mathcal{E})}_{\dot\gamma(t)} s(t) \stackrel{\text{def}}{=} \nabla^{\gamma^*\mathcal{E}}(s)(t)$. The parallel transport map along $\gamma(t)$

$$\tau_\gamma(t) \in \operatorname{Hom}(\mathcal{E}_{\gamma(t)}, \mathcal{E}_{(t)})$$

is defined by the o.d.e.



$$\boxed{\nabla^{\mathcal{E}}_{\dot\gamma(t)} \underset{\text{matrix}}{\tau_\gamma(t)} = 0.} \qquad \tau_\gamma(0) = I$$

If $\mathcal{E} = \mathcal{E}_{\text{triv}} = M \times \mathbb{C}^n$, $s(t) = (\gamma(t), f(t))$, $f : \mathbb{R}^1 \to \mathbb{R}^n$,

$$\boxed{\nabla^{\mathcal{E}}_{\dot\gamma(t)} s(t) = \left(\gamma(t), \frac{df(t)}{dt} + \vartheta(\gamma(t))f(t)\right)}$$

Then we have

$$(*) \qquad \nabla^{\mathcal{E}}_{\dot\gamma} s(0) = \lim_{t\to 0} \frac{\tau_k(t)s(0) - s(t)}{t}$$

Exercise. Show $(*)$, ie $\lim_{t\to 0} \frac{\tau_k(t)e_i(0)-e_i(t)}{t} = \sum_{j=0}^n \vartheta_{ji}(0)e_j(0)$.

Proposition. *$(\tau_{\gamma_1}\tau_{\gamma_2} - \tau_{\gamma_1^{-1}}\tau_{\gamma_2^{-1}})s(0) = t_1 t_2[\nabla_{\dot\gamma_1}, \nabla_{\dot\gamma_2}]s(0) = t_1 t_2 \Omega(\dot\gamma_1, \dot\gamma_2)s(0)$, with $\tau_{\gamma_i}(t_i)s(0) = s(t) + t\nabla_{\dot\gamma_i} s(0)$.*

Proof. Exercise.

Thus we see curvature is the infinitesimal version of holonomy.

Let us consider the non-trivial 1-manifold S^1 with (oriented) complex line bundle $\mathcal{L}$ (necessarily trivial) with metric connection ∇ (ie parallel transport preserves the metric) and $\mathcal{M}$ = gauge equiv. classes of unitary connections:

Proposition. *(Ahranov-Bohm effect) $\mathcal{M}(S^1) \cong U(1)$. (More precisely: Gauge equivalence classes of unitary connections on $\mathcal{L} \to S^1$ are parameterized by S^1).*

More generally, for an (oriented) (trivial) complex n-bundle $\mathcal{E} \xrightarrow{\nabla} S^1$, with ∇ unitary, so that

$$(X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle),$$

one has

$$\boxed{\mathcal{M}(S^1; \mathcal{E}) \cong U(n)}$$

Equivalently, $\mathcal{M}(S^1; \mathcal{E})$ is in 1-1 correspondence with self-adjoint boundary conditions for Dirac-type operator

$$D = i\nabla = i\frac{d}{dx} + \underset{\text{hermitian}}{A(x)} \quad : C^\infty([0, 2\pi]; \mathcal{E}_{\text{triv}}) \to C^\infty([0, 2\pi]; \mathcal{E}_{\text{triv}})$$

($\mathcal{E}_{\text{triv}} \cong S^1 \times \mathbb{C}^n$) over $[0, 2\pi]$, with respect to the inner-product

$$\langle s_1, s_2 \rangle = \int_0^{2\pi} \bar{s}_1(t) s_2(t)\, dt$$

A boundary condition for D means a subspace $\omega \subset \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^{2n}$. That is, $\omega = \omega_0 \oplus \omega_{2\pi}$ where ω_0, $\omega_{2\pi}$ are subspaces of the fibres of $\mathcal{E}$ over the boundary components. Thus the space of <u>all</u> boundary conditions is the full Grassmanian $\mathrm{Gr}(\mathbb{C}^{2n}) = \bigcup_{k=1}^{2n} \mathrm{Gr}_k(\mathbb{C}^{2n})$.

That brings us to $0 + 1$-dimensional topological quantum field theory (TQFT).

3.3 Geometric Quantization.

Let us very quickly first review HAMILTONIAN (CLASSICAL) MECHANICS.

Let M be a $2n$-dimensional symplectic manifold. (M =phase space of classical system).

M has 2-form $\omega \in \Omega^2(M)$, $d\omega = 0$ and $\omega^n \neq 0$.

$\omega^n \neq 0 \iff$ non-degenerate $\iff \omega : TM \to TM^*$ is invertible with inverse $\omega^{-1} : T^*M \to TM$,

$$\omega = \omega_{ij} dx_i \wedge dx_j \quad \text{locally}$$

$$\omega^{-1} = \omega^{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}, \qquad \omega^{ij}\omega_{jk} = \delta_{ik}$$

ω relates classical observables $f \in C^\infty(M)$ to vector fields $X_f \in C^\infty(M; TM)$ by

$$X_f = \omega^{-1}(df) \qquad (\text{or } df = -i_X\omega)$$

This is a symplectic vector field, ie

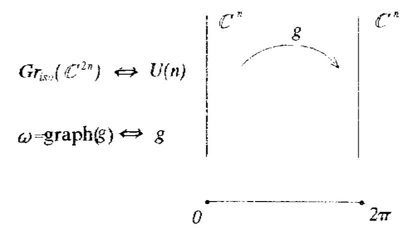
$$(*) \qquad \mathcal{L}_{X_f}\omega = 0 \qquad (\text{Lie derivative}).$$

(There is a partial converse to $(*)$: $\mathcal{L}_X\omega = 0 \implies i_X\omega = df$ some $f \in C^\infty(M)$ <u>locally</u> since $f_X = i_X \circ d + d \circ i_X$ and $d\omega = 0$).

In local canonical coords. $\omega = dp_i \wedge dq_i$ and

$$df = \frac{\partial f}{\partial p_i} dq_i - \frac{\partial f}{\partial q_i} dq_i.$$

The <u>flow</u> of the vector field X_f is given by



$$\frac{\partial q_i}{\partial t} = \frac{\partial f}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial f}{\partial q_i} \qquad \text{(Hamilton's equations)}.$$

flow $\rho(t) = (p(t), q(t))$.

Quantization.
Ideally quantization of a classical mechanical system with symplectic phase (M,ω) would be given by unitary Hilbert space representation of the Poisson Lie algebra of classical observables $C^\infty(M)$, where the Poisson brachet is defined by

$$\{f, g\} = \omega^{-1}(df, dg) = X_f(g),$$

so one looks for a Hilbert space $\mathcal{H}$ with algebra $\mathcal{A}$ of self adjoint operators, and a linear map

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \mathcal{A} \\ f & \longmapsto & \hat{f} \end{array}$$

such that $\widehat{\{f,g\}} = ih[\hat{f}, \hat{g}]$ and f const $\implies \hat{f}$ mult$^{\underline{n}}$ operator.

If there exists complex $U(1)$-line bundle $\mathcal{L} \to M$ with connection ∇ such that

$$\boxed{\Omega^{(\nabla)} = i\omega}$$

then (M,ω) is 'pre-quantizable' with

$$\boxed{H = L^2(M; \mathcal{L}) \quad f \longmapsto \hat{f} = -ih\nabla_{X_f} + f}$$

To get a good quantization we need a polarization. Roughly this means restricting the quantization to those sections annihilated by half the derivatives on M. Equivalently, it means modifying the map $f \to \hat{f}$ to take account of the commutative assosiative algebra structure defined on $C^\infty(M)$ by pointwise multiplication of functions, in addition to respecting the Poisson algebra structure. That is, we also require $\widehat{fg} = \hat{f} \cdot \hat{g}$.

This restricts greatly the quantizable classical observables. If M is a Kahler manifold there is a natural polarization given by restricting to those sections such that $\nabla^{0,1}s = (I + iJ)\nabla s = 0$, where $J \in C^\infty(M; \operatorname{End} TM)$ is the complex structure on M $(J^2 = -1)$.

$$\boxed{\mathcal{H}_{\text{pol}} = \Gamma_{\text{hol}}(M; \mathcal{L})}$$

Example. $M = \mathbb{C}^n$, $\mathcal{L} = M \times \mathbb{C}$, $\omega = \frac{i}{2} dz_i \wedge d\bar{z}_i = dp_i \wedge dqi = i\partial\bar{\partial}K$, where K is Kahler potential.

$\mathcal{H}_{\text{p}} = \operatorname{Hol}(\mathbb{C}^n; \mathbb{C})$ with

$$\langle \psi, \psi' \rangle = \int_{\mathbb{C}^n} e^{-\frac{1}{2h}\sum |z_i|^2} \bar{\phi}\phi' \omega^n \qquad (d\vartheta = \Omega = i\omega)$$

As an exercise, the reader might like to check that under the gauge transformation $g(z) = e^{-iK(z,\bar{z})}$ $(K(z,\bar{z}) = \frac{1}{2}\sum_i |z_i|^2)$ the holomorphic sections of $\mathcal{L}$ now take the form $\psi(z,\bar{z}) = \phi(z) e^{-\frac{1}{4h}\sum |z_i|^2}$ (where the ϕ are holomorphic functions in the usual sense) with ∇ potential $\vartheta = -\frac{i}{2} z^i dz^i + \frac{i}{2} dK$.

Harmonic oscillator $h(z,\bar{z}) = \sum_i \frac{|z_i|^2}{2}$ quantizes to $\hat{h} = \sum h z_i \frac{\partial}{\partial z} + \frac{h}{2}$,

$$\hat{h}\phi = \left(h \underbrace{z}_{\text{'creation operator'}} \underbrace{\frac{\partial}{\partial z}}_{\text{'annihilation operator'}} + \frac{h}{2} \right) \phi.$$

We may use Kahler quantization to define a $TQFT$ in dim 1 as follows.

Take Lagrangian $\mathcal{L}(\psi) = \int_X \bar{\psi} D\psi \, d\psi$, defined as a functional. $\mathcal{L} : C^\infty(X; \mathcal{E}^{\mathbb{C}}_{\text{triv}}) \to \mathbb{R}$, when $\mathcal{E}^{\mathbb{C}}_{\text{triv}}$ is a Hermitian n-bundle over $[0, 2\pi]$ (necessarily trivial) with metric connection ∇, and $D = i\nabla_{\frac{d}{dx}}$ is the associated Dirac operator.

Define QFT over phase space $\operatorname{Gr}(\mathbb{C}^{2n}) = \bigcup_k \operatorname{Gr}_k(\mathbb{C}^n)$ by the Feynman path integral

$$Z_X(\omega) = \int_{C^\infty_\omega(X; \mathcal{E}_{\text{triv}})} e^{-\mathcal{L}_\omega(\psi)} D\psi \, D\psi^*,$$

where $\mathcal{L}_\omega(\psi) = \int_X \bar{\psi} D_\omega \psi(\omega)$.

Evaluated as a fermionic integral

$$Z_X(\omega) \sim \det D_\omega.$$

Relative to a trivialization of $\mathcal{E}^{\mathbb{C}}_{\text{triv}} \cong [0, 2\pi] \times \mathbb{C}^n$ we may write $D = i\frac{d}{dx} + A(x) : C^\infty([0, 2\pi], \mathbb{C}^n) \to C^\infty([0, 2\pi], \mathbb{C}^n)$.

But $\det D : \operatorname{Gr}(\mathbb{C}^{2n}) \to ?$ is not a function but a holomorphic section of a complex line bundle over $\operatorname{Gr}(\mathbb{C}^{2n})$ with fibre

$$\mathcal{L}_\omega = \operatorname{Det}(\operatorname{Ker} D_\omega)^* \otimes \operatorname{Det}(\operatorname{CoKer} D_\omega) \qquad (\text{recall } \operatorname{Det} V = \bigwedge\nolimits^{\max} V)$$

Since D_ω is Fredhow, Ker D_ω and CoKer D_ω are finite dimensional and so $\mathcal{L}_\omega$ is well defined. One finds

$$\mathcal{L} \cong \mathrm{Det}(\mathcal{E}).$$

The tautological bundle over Gr $(\mathbb{C}^{2n})$ we considered earlier. So because Gr $(\mathbb{C}^{2n})$ is a Kahler manifold and $\mathcal{L}$ has a canonical connection ∇ with*

$$\Omega^{(\nabla)} = i\omega \qquad (\omega = \text{Kahler form})$$

then we can geometrically quantize. We obtain

$$\mathcal{H} = \Gamma_{\text{hol}}(\text{Gr } (\mathbb{C}^{2n}); \mathcal{L}^*)^* \qquad (\text{Fock space})$$
$$\cong \bigwedge \mathbb{C}^{2n} = \bigoplus_k \bigwedge^k \mathbb{C}^{2n}$$

—which is the fundamental irreducible representation of the group $U(2n)$, (Note $\mathrm{Gr}_k(\mathbb{C}^{2n}) \cong U(2n)/U(k)\times U(n-k)$.) And that, very roughly, is why 0+1-dimensional topological quantum field theory is essentially equivalent to the representation theory of the compact Lie groups, and hence to quantum mechanics.

*See for example R. O. Wells, '*Complex Manifolds*' (Springer-Verlag) or P. Griffiths, J. Harris, '*Principles of Algebraic Geometry*' (Wiley).

TENSORES Y GEOMETRÍA DIFERENCIAL

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Resumen: Vamos a explotar todas las posibilidades de la diferencial de una función. Con ella fabricaremos la base dual del espacio cotangente y su espacio tensorial. Por dualidad, extenderemos nuestra presentación, didáctica, al espacio tangente y a sus tensores.

1. Introducción

Se trata de dar una visión pedagógica de los espacios tensoriales con ayuda de las diferenciales de las variables independientes. Este esquema pedagógico permite ir directamente al corazón de la geometría diferencial sin seguir el proceso tradicional. A nivel introductorio siempre se ha considerado que la diferencial dx se comporta como algo "pequeño" y constante, a diferencia del incremento Δx que es variable y tiende a cero.

Vamos a considerar dx, dy, dz, etc, como funciones, más exactamente, como formas lineales o tensores covariantes de rango uno. La diferencial df de una función real de variable real f, será también una forma lineal o tensor. df será una combinación lineal de las diferenciales dx, dy, etc. Este aspecto algebraico nos conducirá a los tensores. No debe olvidarse que df es la aproximación lineal de la función f la cual es, en general, no lineal. Según esta visión, se reemplaza un pedacito de la curva representativa de f (o de su superficie representativa, cuando se tienen varias variables) por un pedacito de línea recta.

Esto quiere decir que localmente f se comporta como una recta, la línea recta tangente (o el plano tangente en el caso de una superficie).

Al referirnos a las propiedades locales de curvas y superficies, estamos tocando las propiedades que justamente estudia la geometría diferencial en su aspecto clásico.

Las otras propiedades: curvatura, torsión, triedro móvil, conexión, etc, no las tomaremos en consideración. Sólo nos contentaremos con explorar la tangente a una curva, el plano tangente o el espacio tangente a una variedad, en general. Además, lo que nos interesa es el aspecto algebraico, la estructura analítica la dejaremos de lado.

Primer Encuentro de Geometría Diferencial en Física.
Marzo 3-5, 1994. Universidad de los Andes.